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Fourier Analysis for Navier-Stokes Equations

Mater's thesis submitted in partial fullfilment of the requirements for the degree of Master of Science in Technology

Espoo, 21.12.2007

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ABSTRACT OF MASTER'S THESIS

18.12.2007

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Department:	Department of Engineering Physics and Mathematics
Major subject:	Mathematics
Minor subjects:	Information technology
English title:	Fourier Analysis for Navier-Stokes Equations
Finnish title:	Fourier-analyysi Navier-Stokes -yhtälöille
Number of pages:	109
Chair:	Mat-1 (Mathematics)
Supervisor:	prof. Timo Eirola
Instructor(s):	lect. Pekka Alestalo
Abstract:	This thesis introduces a Fourier analysis for incompressible three-dimensional Navier-Stokes partial differential equation system. The Fourier transform, that is applied, is defined by Riemann-integral.
Keywords:	Fourier Analysis, Navier-Stokes Equations, Fourier inversion theorem
Study secretary fills:	
	Thesis approved:
	Library code:

Preface

This thesis is written as a consequence of my academic approach to studies in Helsinki University of Technology. The laboratory of mathematics, that has an axiomatic approach to Fourier and Laplace methods has been a fine place to make a scientific document, this thesis. I thank the professors and teachers in our laboratory, especially them, who have provided me material and knowledge on the topic. I also thank all other scientists on the internet, who have released material on the topic because it has been useful for me regardless the official publication status or the force of mathematical proof. I also thank my parents, sister and brother on the understanding attitude they have taken to the slow advancement, that I found the most useful way to finish the work.

Juha-Matti Vihtanen

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Introduction

The motivation of this thesis originates from my mathematics undergraduate studies. The study material included existence theorems that were written for linear ordinary differential equations. The strategy was to study foundations of mathematics so that it is easy to interpret, analyze and solve natural laws. Partial differential equations, that describe natural phenomena mathematically, have an important role in the study of natural sciences.

It is known that the heat equation has an explicit solution formula and that Maxwell equations are linear and electromagnetic waves transversal. However, properties of Navier-Stokes equations were almost completely beyond my horizon. I had learned that many natural laws can be formulated using differential operators, that have eigenfunctions and eigenvalues.

Dispersitivity was problem to me. I knew that Maxwell equations may have singular solutions. I thought that they might be too difficult to handle mathematically and hence research on existence of solutions of Navier-Stokes equations might be a better approach to mathematical analysis of natural laws.

Some well-known existence theorems, that are written for solutions of ordinary differential equations, are based on uniform convergence. Picard-Lindelöf -iteration method is an example. It is also known that engineers apply Laplace and Fourier transform methods to solve differential and partial differential equations. I decided to apply the system, that is, to transform the equation system, solve it in Fourier-domain and inverse transform the solution. I thought that if the strategy is successful, it proves the existence and uniqueness of solutions of Navier-Stokes equations.

My aim was to show Fourier inversion theorem. For that purpose I needed a suitable topic for my thesis. I thought that a sufficient condition for getting such a topic is to find an application that is based on the use of a convolution formula, that applies separation of convolution and Fourier inversion theorem. One of the professors of our university said during a lecture, that someone could try to show the existence of solutions of Navier-Stokes equations, because the research in our university is centered mainly around analysis. I noticed that Navier-Stokes equations contain a product term, that will be transformed into convolution. I was quite sure that Fourier inversion theorem would be applied in some stage of the analysis. After some unsuccesful trials, I finally succeeded to find a supervisor.

I planned to transform the equation system, apply the convolution formula mentioned above, iterate a solution using properties of the transfer function of the subsidiary equation and inverse transform the limit to obtain a solution. The first two steps succeeded, but, unfortunately, the iteration was not successful and hence I did not succeed to show the existence and uniqueness of solutions of Navier-Stokes equations. Instead of that, I found a subsidiary equation and estimates.

The proof of the convolution formula applies Fourier inversion theorem. I noticed accidentally that my study material did not contain any version of Fourier inversion theorem with suitable assumptions and hence gained the possibility to show it myself. I found assumptions that fit to the problem but covered also some important examples of plain transfer functions.

Chapter 1

Some Fourier analysis tools

1.1 Definitions and lemmas

Definition 1.1. The set of all functions $f : \mathbb{R} \to \mathbb{C}$ is X. The set of all scalar fields $x : \mathbb{R}^3 \to \mathbb{C}$ is H. The set $\mathbb{R}^3 \setminus \{0\}^3$ is S. The set of all scalar fields $x : S \to \mathbb{C}$ is V. The set of all complex numbers z with $\mathrm{Im}(z) < 0$ is Π^-. The set of all complex numbers z with $\mathrm{Im}(z) > 0$ is Π^+. The set $S \times \overline{\Pi^-}$ is Z. The set of all scalar fields $f : S \times \mathbb{R} \to \mathbb{C}$ is W.

Definition 1.2. Assume $n \in \mathbb{N}$. Arithmetic operation $\cdot : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is defined by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

The operation $\cdot$ is a commutative product operation and it is called dot product.

Lemma 1.3. *For dot product* $\cdot : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ *we have the inequality*

$$|x \cdot y| \leq |x||y|.$$

Proof: We calculate

$$\begin{aligned} |x \cdot y| &= \left|\sum_{i=1}^{n} x_i y_i\right| \leq \sum_{i=1}^{n} |x_i y_i| = \sum_{i=1}^{n} |x_i||y_i| \leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |y_i|^2\right]^{\frac{1}{2}} \\ &= |x||y| \end{aligned}$$

and obtain the claim. □

The inequality of the Lemma 1.3 is called Schwartz inequality.

Definition 1.4. We define scalar product $\cdot$ from $\mathbb{C}^n \times \mathbb{R}$ to $\mathbb{C}^n$ by

$$[x \cdot \alpha]_k = [x]_k \cdot \alpha$$

For notational convenience we write $x\alpha$ instead of $x \cdot \alpha$.

Lemma 1.5. *We have the equation*

$$x\alpha = \alpha x$$

Proof: We calculate

$$[x\alpha]_k = x_k\alpha = \alpha x_k = [\alpha x]_k$$

and obtain the claim. □

Definition 1.6. Function $x : \mathbb{R}^n \to \mathbb{C}$ is smooth, if it is infinitely differentiable. The set of all smooth functions is $C^\infty(\mathbb{R}^n)$.

Definition 1.7. The set of all bounded functions $x \in C^\infty(\mathbb{R}^n)$ is $BC^\infty(\mathbb{R}^n)$.

Definition 1.8. The set of all Schwartz test functions on $\mathbb{R}$ is $\mathscr{S}$. The set of all Schwartz test functions on $\mathbb{R}^n$ is $\mathscr{S}_n$.

For further details see [9].

Theorem 1.9. *Assume that function $f : \mathbb{R} \to \mathbb{C}^n$ is continuous. Then we have the equation*

$$f(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt \tag{1.1}$$

for every $x \in \mathbb{R}$.

Proof: Fix $x \in \mathbb{R}$ and $\epsilon > 0$. Fix $\delta > 0$ such that $|f(x') - f(x)| < \frac{\epsilon}{2}$ for every $x' \in (x - \delta, x + \delta)$. Take any h such that $0 < |h| < \delta$. We calculate

$$\begin{aligned}
\left|\frac{1}{h}\int_x^{x+h} f(t)dt - f(x)\right| &= \left|\frac{1}{h}\int_0^h f(x+t)dt - f(x)\right| \\
&= \left|\frac{1}{h}\int_0^h (f(x+t) - f(x))dt\right| \\
&= \left|\frac{1}{h}\int_0^1 (f(x+ht) - f(x))hdt\right| \\
&= \left|\int_0^1 (f(x+ht) - f(x))dt\right| \\
&\leq \int_0^1 |f(x+ht) - f(x)|dt \\
&\leq \int_0^1 \frac{\epsilon}{2}dt = \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

This shows the claim. □

Definition 1.10. *We define*

$$K : \mathbb{R}^3 \to \mathbb{C}, \quad K(x) = \begin{cases} \frac{1}{4\pi}|x|^{-1}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Definition 1.11. *Assume $d = 3$. The set of all functions $f \in C^2(\mathbb{R}^d)$ with bounded absolutely integrable derivatives of all orders (including zero) and estimates*

$$\begin{aligned}
|f(x)| &\leq C|x|^{2-d}, \quad C > 0 \\
|\nabla f(x)| &\leq D|x|^{-2}, \quad D > 0
\end{aligned}$$

is denoted by $\acute{V}$.

Theorem 1.12. *Assume that $f \in \acute{V}$. Assume*

$$u(x) = \int_{\mathbb{R}^3} K(y) f(x-y) dy.$$

Then we have the equation

$$\partial_i^k u(x) = \int_{\mathbb{R}^3} K(y) \partial_i^k f(x-y) dy.$$

for every $k \in \{1,2\}$ for every $i \in \{1,2,3\}$.
Proof: Define

$$\acute{u}_i^k(x) = \int_{\mathbb{R}^3} K(y) \partial_i^k f(x-y) dy$$

for $k \in \{1,2\}$. To show the continuity of $\acute{u}_i^k$, assume $j \in \{1,2,3\}$ and $t \in \mathbb{R}$. We calculate

$$\begin{aligned}
\int_{\mathbb{R}^3} |K(y)\partial_j \partial_i^k f(x-y+te_j)| dy &= \int_{B(0,1)} |K(y)\partial_j \partial_i^k f(x-y+te_j)| dy \\
&\quad + \int_{B(0,1)^{\complement}} |K(y)\partial_j \partial_i^k f(x-y+te_j)| dy \\
&= \int_{B(0,1)} |K(y)||\partial_j \partial_i^k f(x-y+te_j)| dy \\
&\quad + \int_{B(0,1)^{\complement}} |K(y)||\partial_j \partial_i^k f(x-y+te_j)| dy \\
&\leq \int_{B(0,1)} K(y)||\partial_j \partial_i^k f||_\infty dy \\
&\quad + \int_{B(0,1)^{\complement}} \left|\frac{1}{4\pi}\right| |\partial_j \partial_i^k f(x-y+te_j)| dy \\
&\leq ||\partial_j \partial_i^k f||_\infty \int_{B(0,1)} K(y) dy \\
&\quad + \int_{\mathbb{R}^3} \frac{1}{4\pi} |\partial_j \partial_i^k f(x-y+te_j)| dy \\
&= C_{i,j,k} \in \mathbb{R}
\end{aligned}$$

for every $t \in \mathbb{R}$. It is known, that $\mathbb{Z}^3$ is countable. Hence the set $\mathbb{R}^3$ is σ-compact and the space $\mathbb{R}^3$ a σ-finite measure space with Lebesgue measure. The interval $[0,h]$ is clearly a σ-finite measure space with Lebesgue measure. Assumptions imply, that the latter integral in the change of order of integration below is absolutely convergent. We calculate using Fubini's theorem (Theorem 8.8 in [11]) to obtain

$$\begin{aligned}
\acute{u}_i^k(x+he_j) - \acute{u}_i^k(x) &= \int_{\mathbb{R}^3} K(y)\partial_i^k f(x+he_j-y) dy \\
&\quad - \int_{\mathbb{R}^3} K(y)\partial_i^k f(x-y) dy
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} K(y)(\partial_i^k f(x+he_j-y) - \partial_i^k f(x-y))dy \\
&= \int_{\mathbb{R}^3} K(y)(\partial_i^k f(x-y+he_j) - \partial_i^k f(x-y))dy \\
&= \int_{\mathbb{R}^3} K(y)\int_0^h \partial_j\partial_i^k f(x-y+te_j)dtdy \\
&= \int_{\mathbb{R}^3}\int_0^h K(y)\partial_j\partial_i^k f(x-y+te_j)dtdy \\
&= \int_0^h\int_{\mathbb{R}^3} K(y)\partial_j\partial_i^k f(x-y+te_j)dydt.
\end{aligned}
$$

We estimate further and obtain

$$
\begin{aligned}
|\acute{u}_i^k(x+he_j) - \acute{u}_i^k(x)| &= \left|\int_0^h\int_{\mathbb{R}^3} K(y)\partial_j\partial_i^k f(x-y+te_j)dydt\right| \\
&= \left|\int_0^1\int_{\mathbb{R}^3} K(y)\partial_j\partial_i^k f(x-y+hte_j)dyhdt\right| \\
&\leq \int_0^1\left|\int_{\mathbb{R}^3} K(y)\partial_j\partial_i^k f(x-y+hte_j)dyh\right|dt \\
&= \int_0^1\left|\int_{\mathbb{R}^3} K(y)\partial_j\partial_i^k f(x-y+hte_j)dy\right|\left|h\right|dt \\
&\leq \int_0^1\int_{\mathbb{R}^3} |K(y)\partial_j\partial_i^k f(x-y+hte_j)|dy|h|dt \\
&\leq \int_0^1 C_{i,j,k}|h|dt = C_{i,j,k}|h| \to 0
\end{aligned}
$$

as h approactes to 0. This shows that $\acute{u}_i^k$ is continuous with respect to the component j for every $j \in \{1,2,3\}$. Hence $\acute{u}_i^k$ is continuous. Assumptions imply, that the latter integral in the change of order of integration below is absolutely convergent. We calculate using Fubini's theorem (Theorem 8.8 in [11]) and Theorem 1.9 to obtain

$$
\begin{aligned}
u(x+he_i) - u(x) &= \int_{\mathbb{R}^3} K(y)f(x+he_i-y)dy - \int_{\mathbb{R}^3} K(y)f(x-y)dy \\
&= \int_{\mathbb{R}^3} (K(y)f(x+he_i-y) - K(y)f(x-y))dy \\
&= \int_{\mathbb{R}^3} K(y)(f(x+he_i-y) - f(x-y))dy \\
&= \int_{\mathbb{R}^3} K(y)(f(x-y+he_i) - f(x-y))dy \\
&= \int_{\mathbb{R}^3} K(x-y)(f(y+he_i) - f(y))dy \\
&= \int_{\mathbb{R}^3} K(x-y)\int_0^h \partial_i f(y+te_i)dtdy \\
&= \int_{\mathbb{R}^3}\int_0^h K(x-y)\partial_i f(y+te_i)dtdy
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^h \int_{\mathbb{R}^3} K(x-y)\partial_i f(y+te_i)dydt = \int_0^h \int_{\mathbb{R}^3} K(y)\partial_i f(x-y+te_i)dydt \\
&= \int_0^h \int_{\mathbb{R}^3} K(y)\partial_i f(x+te_i-y)dydt = \int_0^h \acute{u}_i^1(x+te_i)dt.
\end{aligned}
$$

We calculate the derivative of u to the direction e_i and obtain

$$
\begin{aligned}
\partial_i u(x) &= \lim_{h\to 0} \frac{u(x+he_i)-u(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_0^h \acute{u}_i^1(x+te_i)dt \\
&= \acute{u}_i^1(x) = \int_{\mathbb{R}^3} K(y)\partial_i f(x-y)dy.
\end{aligned}
$$

In a similar way we obtain

$$
\partial_i^2 u(x) = \int_{\mathbb{R}^3} K(y)\partial_i^2 f(x-y)dy.
$$

This shows the claim. □

Theorem 1.13. *Assume that* $f \in \acute{V}$ *and*

$$
\Delta' f(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} f(x-x')\frac{1}{|x'|}dx'.
$$

Then we have the equation

$$
\Delta\Delta' f \;=\; f
$$

Proof: Define

$$
u(x) = \int_{\mathbb{R}^3} K(x-y)f(y)dy.
$$

According to the definition we have

$$
\begin{aligned}
u(x) &= \int_{\mathbb{R}^3} K(x-y)f(y)dy = \int_{\mathbb{R}^3} K(y)f(x-y)dy \\
&= \int_{\mathbb{R}^3} \frac{1}{4\pi}|y|^{-1}f(x-y)dy \\
&= -\left[-\frac{1}{4\pi}\int_{\mathbb{R}^3} f(x-y)\frac{1}{|y|}dy\right] = -\Delta' f(x).
\end{aligned}
$$

We calculate

$$
\begin{aligned}
\Delta u(x) &= \left[\sum_{i=1}^3 \partial_i^2 u\right](x) = \sum_{i=1}^3 \partial_i^2 u(x) = \sum_{i=1}^3 \int_{\mathbb{R}^3} K(y)\partial_i^2 f(x-y)dy \\
&= \int_{\mathbb{R}^3} \sum_{i=1}^3 K(y)\partial_i^2 f(x-y)dy = \int_{\mathbb{R}^3} K(y)\sum_{i=1}^3 \partial_i^2 f(x-y)dy \\
&= \int_{\mathbb{R}^3} K(y)\left[\sum_{i=1}^3 \partial_i^2 f\right](x-y)dy = \int_{\mathbb{R}^3} K(y)\Delta f(x-y)dy.
\end{aligned}
$$

Fix $\epsilon > 0$. We have $f \in C^2(\overline{B(0,\epsilon)}^{\mathsf{C}})$ and $f \in C^1(B(0,\epsilon)^{\mathsf{C}})$. We also have $K \in C^2(\overline{B(0,\epsilon)}^{\mathsf{C}})$ and $K \in C^1(B(0,\epsilon)^{\mathsf{C}})$. For big values of $|x|$ we have

$$\begin{aligned} |f(x)| &\leq C|x|^{d-2} \\ |\nabla f(x)| &\leq C|x|^{-2} \end{aligned}$$

$$\begin{aligned} |K(x)| &\leq C|x|^{d-2} \\ |\nabla K(x)| &\leq C|x|^{-2}, \end{aligned}$$

where $d = 3$. We calculate using Theorem 14.3 in [1]

$$\begin{aligned} \Delta u(x) &= \int_{\mathbb{R}^3} K(y)\Delta f(x-y)dy \\ &= \int_{\mathbb{R}^3\setminus B(0,\epsilon)} K(y)\Delta f(x-y)dy + \int_{B(0,\epsilon)} K(y)\Delta f(x-y)dy \\ &= \int_{\partial B(0,\epsilon)} K(y)\frac{\partial f}{\partial \nu}(x-y)dy - \int_{\mathbb{R}^3\setminus B(0,\epsilon)} \nabla K(y)\nabla_y f(x-y)dy \\ &\quad + \int_{B(0,\epsilon)} K(y)\Delta f(x-y)dy \end{aligned}$$

$$\begin{aligned} &= \int_{\partial B(0,\epsilon)} K(y)\frac{\partial f}{\partial \nu}(x-y)dy - \int_{\partial B(0,\epsilon)} \frac{\partial K}{\partial \nu}(y)f(x-y)dy \\ &\quad + \int_{\mathbb{R}^3\setminus B(0,\epsilon)} \Delta K(y)f(x-y)dy + \int_{B(0,\epsilon)} K(y)\Delta f(x-y)dy \\ &= I_\epsilon + J_\epsilon + K_\epsilon + L_\epsilon \end{aligned}$$

and analyze terms I_ϵ, J_ϵ, K_ϵ, L_ϵ separately to obtain results

$$\begin{aligned} |I_\epsilon| &= \left|\int_{\partial B(0,\epsilon)} K(y)\frac{\partial f}{\partial \nu}(x-y)dy\right| \leq \int_{\partial B(0,\epsilon)} \left|K(y)\frac{\partial f}{\partial \nu}(x-y)\right|dy \\ &= \int_{\partial B(0,\epsilon)} \left|K(y)\right|\left|\frac{\partial f}{\partial \nu}(x-y)\right|dy \leq \int_{\partial B(0,\epsilon)} \frac{1}{4\pi|y|}||\nabla f||dy \\ &= 4\pi\epsilon^2\frac{1}{4\pi\epsilon}||\nabla f|| = \epsilon||\nabla f|| \to 0, \quad \epsilon \to 0 \end{aligned}$$

$$\begin{aligned} J_\epsilon &= -\int_{\partial B(0,\epsilon)} \frac{\partial K}{\partial \nu}(y)f(x-y)dy = -\int_{\partial B(0,\epsilon)} \nabla K(y)\cdot \nu f(x-y)dy \\ &= -\int_{\partial B(0,\epsilon)} \left[\frac{-1}{4\pi}\frac{y}{|y|^3}\right]\cdot\left[-\frac{y}{|y|}\right]f(x-y)dy \\ &= -\frac{1}{4\pi\epsilon^2}\int_{\partial B(0,\epsilon)} f(x-y)dy \to -f(x), \quad \epsilon \to 0 \end{aligned}$$

$$K_\epsilon = \int_{\mathbb{R}^3\setminus B(0,\epsilon)} \Delta K(y)f(x-y)dy = 0$$

$$\begin{aligned} |L_\epsilon| &= \left|\int_{B(0,\epsilon)} K(y)\Delta f(x-y)dy\right| \leq \int_{B(0,\epsilon)} \left|K(y)\Delta f(x-y)\right|dy \\ &= \int_{B(0,\epsilon)} \left|K(y)\right|\left|\Delta f(x-y)\right|dy = \int_{B(0,\epsilon)} \frac{1}{4\pi|y|}||\Delta f||dy \\ &= \int_0^\epsilon \frac{1}{4\pi r}||\Delta f||4\pi r^2 dr = \frac{1}{2}\epsilon^2||\Delta f|| \to 0, \quad \epsilon \to 0. \end{aligned}$$

Hence

$$\Delta u(x) = -f(x)$$

and we have

$$\begin{aligned} f(x) &= -\Delta u(x) = -\Delta(-\Delta' f)(x) = \Delta\Delta' f(x) \\ f &= \Delta\Delta' f. \end{aligned}$$

This shows the claim. □

Corollary 1.14. *The Laplacian $\Delta : \Delta'\acute{V} \to H$ has right inverse*

$$\Delta^{-1} : \acute{V} \to \Delta'\acute{V}, \quad \Delta^{-1}u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} u(x-x')\frac{1}{|x'|}dx'.$$

Proof: Fix $u \in \acute{V}$. We define

$$\Delta' u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} u(x-x')\frac{1}{|x'|}dx'$$

and calculate

$$(\Delta\Delta')u = \Delta\Delta' u = u = Iu.$$

Hence

$$(\Delta\Delta')u = Iu$$

for every $u \in \acute{V}$ and we obtain

$$\Delta\Delta' = I.$$

This shows the claim. □

Definition 1.15. We define function

$$\mu : \mathbb{R} \to \mathbb{C}, \quad \mu(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}.$$

The function μ is step function.

Definition 1.16. We define operator

$$\tau_\alpha : X \to X, \quad \tau_\alpha x(t) = x(t-\alpha).$$

The operator τ_α is translation.

Lemma 1.17. *Translation τ_α is linear and bounded.*
Proof: We calculate

$$\begin{aligned}
\tau_\alpha(x+y)(t) &= (x+y)(t-\alpha) = x(t-\alpha) + y(t-\alpha) = \tau_\alpha x(t) + \tau_\alpha y(t) \\
&= (\tau_\alpha x + \tau_\alpha y)(t) \\
\tau_\alpha(x+y) &= \tau_\alpha x + \tau_\alpha y
\end{aligned}$$

$$\begin{aligned}
\tau_\alpha(cx)(t) &= (cx)(t-\alpha) = cx(t-\alpha) = c\tau_\alpha x(t) = (c\tau_\alpha x)(t) \\
\tau_\alpha(cx) &= c\tau_\alpha x
\end{aligned}$$

$$\begin{aligned}
||\tau_\alpha x|| &= \sup_{t\in\mathbb{R}} |\tau_\alpha x(t)| = \sup_{t\in\mathbb{R}} |x(t-\alpha)| = \sup_{t\in\mathbb{R}} |x(t)| = ||x|| \\
||\tau_\alpha|| &= \sup_{||x||=1} ||\tau_\alpha x|| = \sup_{||x||=1} ||x|| = \sup_{||x||=1} 1 = 1.
\end{aligned}$$

This shows the claim. □

Definition 1.18. We define a special function with the name sinc. The definition is given by sinc : $\mathbb{R} \to \mathbb{C}$,

$$\operatorname{sinc}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega t} d\omega.$$

The function sinc is called sine cardinal.

Lemma 1.19. *We have the equation*

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

for every $t \neq 0$.
Proof: Assume $t \neq 0$. We calculate

$$\operatorname{sinc}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega t} d\omega = \frac{1}{2\pi}\Big/_{-\pi}^{\pi} \frac{1}{it} e^{i\omega t} = \frac{1}{2\pi}\frac{1}{it}(e^{i\pi t} - e^{-i\pi t}) = \frac{\sin(\pi t)}{\pi t}.$$

This shows the claim. □

Lemma 1.20. *The function* sinc *is continuous.*
Proof: Assume $x \neq 0$. Then sinc is continuous at x by Lemma 1.19. Assume $x = 0$. We calculate using Lemma 1.19 to obtain

$$\operatorname{sinc}(x) = \operatorname{sinc}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega 0} d\omega = 1 = \lim_{t\to 0}\frac{\sin(\pi t)}{\pi t} = \lim_{t\to x}\operatorname{sinc}(t).$$

Hence sinc is continuous at x. This completes the proof. □

Lemma 1.21. *We have the equation*

$$\operatorname{sinc}(-t) = \operatorname{sinc}(t)$$

for every $t \in \mathbb{R}$.
Proof: We calculate

$$\begin{aligned}
\operatorname{sinc}(-t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega(-t)} d\omega = \frac{1}{2\pi}\int_{\pi}^{-\pi} e^{i(-\omega)(-t)}(-d\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega t} d\omega \\
&= \operatorname{sinc}(t)
\end{aligned}$$

to obtain the claim. □

Theorem 1.22. *(Cauchy uniform convergence criterion for integrals) Assume that $a \in \mathbb{R}$ and $f \in X$ and $I \subset \mathbb{R}$ is a finite interval. Then the integral*

$$\int_a^\infty f(x,t)dt \tag{1.2}$$

converges uniformly to a limit $h \in C(I)$ if and only if

$$\forall \epsilon > 0 \quad \exists M > 0 \ \text{ s.t. } \ M_1, M_2 > M \ \Rightarrow \ \left| \int_{M_1}^{M_2} f(x,t)dt \right| < \epsilon \quad \forall x \in I.$$

Proof: "$\Rightarrow$" Assume that the integral (1.2) converges uniformly to a limit $h \in C(I)$. Fix $\epsilon > 0$. Set K_1 such that

$$\left| \int_a^{M_1} f(x,t)dt - \int_a^\infty f(x,t)dt \right| \leq \frac{\epsilon}{2}, \quad \forall x \in I,$$

whenever $M_1 > K_1$. Set K_2 such that

$$\left| \int_a^{M_2} f(x,t)dt - \int_a^\infty f(x,t)dt \right| \leq \frac{\epsilon}{2}, \quad \forall x \in I,$$

whenever $M_2 > K_2$. Set $M = \max\{K_1, K_2\}$. Take $M_1 > M$ and $M_2 > M$. We have $M_1 > M \geq K_1$ and $M_2 > M \geq K_2$. We calculate

$$\begin{aligned}
\left| \int_{M_1}^{M_2} f(x,t)dt \right| &= \left| \int_{M_1}^{a} f(x,t)dt + \int_a^{M_2} f(x,t)dt \right| \\
&= \left| - \int_a^{M_1} f(x,t)dt + \int_a^\infty f(x,t)dt \right. \\
&\quad \left. + \int_a^{M_2} x(t)dt - \int_a^\infty f(x,t)dt \right| \\
&\leq \left| \int_a^{M_1} f(x,t)dt - \int_a^\infty f(x,t)dt \right| \\
&\quad + \left| \int_a^{M_2} f(x,t)dt - \int_a^\infty f(x,t)dt \right| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall x \in I
\end{aligned}$$

and obtain the forward implication.

"$\Leftarrow$" Assume that

$$\forall \epsilon > 0 \quad \exists M > 0 \ \text{ s.t } \ M_1, M_2 > M \ \Rightarrow \ \left| \int_{M_1}^{M_2} f(x,t)dt \right| < \epsilon, \quad \forall x \in I.$$

Fix $\epsilon > 0$. Set $e_k = 2^{-(k+1)}$. Take a sequence K_n, $n \in \mathbb{N}$ such that $K_n < K_{n+1}$, $n \in \mathbb{N}$ and $K_n \to \infty$, $n \to \infty$. Take a subsequence K_{n_k} such that

$$M_{1,k}, M_{2,k} > K_{n_k} \ \Rightarrow \ \left| \int_{M_{1,k}}^{M_{2,k}} f(x,t)dt \right| < e_{k+1}, \quad \forall x \in I.$$

Define

$$c_k(x) = \int_a^{K_{n_k}} f(x,t)dt, \quad x \in I.$$

Fix $\acute{\epsilon} > 0$. Fix $N \in \mathbb{N}$ such that $2^{-N} < \acute{\epsilon}$. Take $m, n > N$. We calculate

$$\begin{aligned}
|(c_n - c_m)(x)| &= |c_n(x) - c_m(x)| = \left| \sum_{j=m+1}^{n} c_j(x) - c_{j-1}(x) \right| \\
&= \left| \sum_{j=m+1}^{n} \int_a^{K_{n_j}} f(x,t)dt - \int_a^{K_{n_{j-1}}} f(x,t)dt \right| \\
&= \left| \sum_{j=m+1}^{n} \int_{K_{n_{j-1}}}^{a} f(x,t)dt + \int_a^{K_{n_j}} f(x,t)dt \right| \\
&= \left| \sum_{j=m+1}^{n} \int_{K_{n_{j-1}}}^{K_{n_j}} f(x,t)dt \right| \le \sum_{j=m+1}^{n} \left| \int_{K_{n_{j-1}}}^{K_{n_j}} f(x,t)dt \right| \\
&\le \sum_{j=m+1}^{n} e_{j-1} = \sum_{j=m+1}^{n} 2^{-j} \le 2^{-m} \le 2^{-N} < \acute{\epsilon}, \quad \forall x \in I \\
||c_n - c_m|| &= \sup_{x \in I} |(c_n - c_m)(x)| \le \sup_{x \in I} \acute{\epsilon} = \acute{\epsilon}.
\end{aligned}$$

This shows that the sequence c_k is a Cauchy sequence. Theorem 1.5-5 in [7] shows that $C(I)$ is complete. Hence c_k converges uniformly to a continuous limit h. Fix $k \in \mathbb{Z}_+$ such that

$$\left| \int_c^{K_{n_k}} f(x,t)dt - h(x) \right| \le \frac{\epsilon}{2}, \quad \forall x \in I$$

and $2^{-(k+1)} < \frac{\epsilon}{2}$. Take any $M > K_{n_k}$. We calculate

$$\begin{aligned}
\left| \int_a^{M} f(x,t)dt - h(x) \right| &= \left| \int_a^{K_{n_k}} f(x,t)dt - h(x) + \int_{K_{n_k}}^{M} f(x,t)dt \right| \\
&\le \left| \int_a^{K_{n_k}} f(x,t)dt - h(x) \right| + \left| \int_{K_{n_k}}^{M} f(x,t)dt \right| \\
&\le \frac{\epsilon}{2} + e_k = \frac{\epsilon}{2} + 2^{-(k+1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall x \in I.
\end{aligned}$$

This shows that the integral (1.2) converges uniformly to the limit h and completes the proof. □

Lemma 1.23. *Assume continuous functions $x_M : [a,b] \to \mathbb{C}$ and a function $x : [a,b] \to \mathbb{C}$ such that $x = \lim_{M\to\infty} x_M$ where the convergence is uniform. Then we have the equation*

$$\int_a^b \lim_{M\to\infty} x_M(t)dt = \lim_{M\to\infty} \int_a^b x_M(t)dt.$$

Proof: We have

$$x = \lim_{M\to\infty} x_M,$$

where M is a continuous-valued variable. Take any $\epsilon > 0$. There is $M_0 > 0$ s.t. $||x - x_M|| < \epsilon$ for every $M \in \mathbb{R}$, $M > M_0$. Set $N_0 = \lceil M_0 \rceil$. Then for every $N \in \mathbb{N}$, $N > N_0$ we have $||x - x_N|| < \epsilon$, that is

$$x = \lim_{N\to\infty} x_N,$$

where N is integer-valued. Theorem 7.12 in [10] implies, that the function x is continuous as a limit of a uniformly convergent sequence of continuous functions. Hence x is Riemann-integrable. Assume that $M \to \infty$. We calculate

$$\begin{aligned}
\left|\int_a^b x_M(t)dt - \int_a^b \lim_{M\to\infty} x_M(t)dt\right| &= \left|\int_a^b x_M(t)dt - \int_a^b x(t)dt\right| \\
&= \left|\int_a^b (x_M(t) - x(t))dt\right| \\
&= \left|\int_a^b (x_M - x)(t)dt\right| \\
&= \int_a^b |(x_M - x)(t)|dt \\
&= \int_a^b |x_M - x|(t)dt
\end{aligned}$$

$$\leq \int_a^b \sup_{t\in[a,b]} |x_M - x|(t)dt = \int_a^b ||x_M - x||dt = (b-a)||x_M - x|| \to 0.$$

This shows the claim. $\square$

Lemma 1.24. *We have the equation*

$$\int_0^M \int_0^\infty e^{-tx}(e^{-it} - e^{it})\, dx\, dt = \int_0^\infty \int_0^M e^{-tx}(e^{-it} - e^{it})\, dt\, dx. \tag{1.3}$$

Proof: To show the claim we define

$$I = \int_0^M \int_0^\infty |e^{-tx}(e^{-it} - e^{it})|\, dx\, dt$$

and calculate

$$\begin{aligned}
I &= \int_0^M \int_0^\infty |e^{-tx}(e^{-it} - e^{it})|\, dx\, dt = \int_0^M \int_0^\infty |e^{-tx}||e^{-it} - e^{it}|\, dx\, dt \\
&= \int_0^M \int_0^\infty |e^{-tx}|\, dx |e^{-it} - e^{it}|\, dt = \int_0^M \left|\frac{1}{t}\right|\left| - \Big/_{-1}^{1} e^{it\omega}\right| dt \\
&= \int_0^M \left|\Big/_{-1}^{1} \frac{1}{it} e^{it\omega}\right| dt = \int_0^M \left|\int_{-1}^1 e^{it\omega} d\omega\right| dt \leq \int_0^M \int_{-1}^1 |e^{it\omega}| d\omega dt \\
&= \int_0^M \int_{-1}^1 1 d\omega dt = \int_0^M 2dt = 2M \in \mathbb{R}.
\end{aligned}$$

This shows that the left hand side integral of the equation (1.3) converges absolutely. Recall that Fubini's theorem is formulated for Lebesgue integral. Real numbers $\mathbb{R}$ is a σ-compact set because it is a countable union of compact intervals. Hence it is also a σ-finite mesaure space with Lebesgue measure. Continuity of the exponential function implies, that the function $f(x,t) = e^{-tx}(e^{-it} - e^{it})$ is continuous on $\mathbb{R}^2$ and hence measurable on $\mathbb{R}^2$. Order of integration can be now changed using Fubini's theorem (Theorem 8.8 in [11]). This shows the claim. □

Lemma 1.25. *We have the equation*

$$\int_0^M \operatorname{sinc}(t)\,dt = \frac{1}{\pi}\int_0^\infty \frac{1}{1+x^2}\,dx + \frac{1}{2\pi i}\int_0^\infty \left[\frac{e^{(-x+i)\pi M}}{-x+i} - \frac{e^{(-x-i)\pi M}}{-x-i}\right] dx.$$

Proof: We define

$$I = \int_0^M \operatorname{sinc}(t)\,dt$$

and calculate applying Lemma (1.24) to obtain

$$\begin{aligned}
I &= \int_0^M \operatorname{sinc}(t)\,dt = \int_0^M \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega t} d\omega\, dt = \frac{1}{2\pi}\int_0^M \int_{-\pi}^{\pi} e^{i\omega t} d\omega\, dt \\
&= \frac{1}{2\pi}\int_0^M \int_{-1}^{1} e^{i\pi\omega t}\pi d\omega\, dt = \frac{1}{2\pi}\int_0^{\pi M} \int_{-1}^{1} e^{i\omega t} d\omega\, dt \\
&= \frac{1}{2\pi}\int_0^{\pi M} \Big/_{-1}^{1} \frac{e^{i\omega t}}{it} dt = \frac{1}{2\pi}\int_0^{\pi M} \frac{1}{it}(e^{it} - e^{-it})\,dt \\
&= \frac{1}{2\pi i}\int_0^{\pi M}\int_0^\infty e^{-tx}\,dx\,(e^{it} - e^{-it})\,dt \\
&= \frac{1}{2\pi i}\int_0^{\pi M}\int_0^\infty e^{-tx}(e^{it} - e^{-it})\,dx\,dt \\
&= \frac{1}{2\pi i}\int_0^{\infty}\int_0^{\pi M} e^{-tx}(e^{it} - e^{-it})\,dt\,dx \\
&= \frac{1}{2\pi i}\int_0^{\infty}\int_0^{\pi M} (e^{(-x+i)t} - e^{(-x-i)t})\,dt\,dx \\
&= \frac{1}{2\pi i}\int_0^{\infty}\Big/_0^{\pi M} \left[\frac{e^{(-x+i)t}}{-x+i} - \frac{e^{(-x-i)t}}{-x-i}\right] dx \\
&= \frac{1}{2\pi i}\int_0^{\infty} \left[\frac{e^{(-x+i)\pi M}}{-x+i} - \frac{e^{(-x-i)\pi M}}{-x-i} - \left[\frac{1}{-x+i} - \frac{1}{-x-i}\right]\right] dx \\
&= \frac{1}{\pi}\int_0^\infty \frac{1}{1+x^2}\,dx + \frac{1}{2\pi i}\int_0^\infty \left[\frac{e^{(-x+i)\pi M}}{-x+i} - \frac{e^{(-x-i)\pi M}}{-x-i}\right] dx.
\end{aligned}$$

The definition of I now implies the claim. □

Lemma 1.26. *We have equations*

$$\begin{aligned}
\int_0^\infty \operatorname{sinc}(t)dt &= \frac{1}{2} \\
\int_{-\infty}^0 \operatorname{sinc}(t)dt &= \frac{1}{2}.
\end{aligned}$$

Proof: Assume that $M \to \infty$. We estimate

$$\begin{aligned}
\left|\int_0^M \operatorname{sinc}(t)\,dt - \frac{1}{2}\right| &= \left|\int_0^M \operatorname{sinc}(t)\,dt - \frac{1}{\pi}\Big/_0^\infty \arctan t\right| \\
&= \left|\int_0^M \operatorname{sinc}(t)\,dt - \frac{1}{\pi}\int_0^\infty \frac{1}{1+t^2}\,dt\right| \\
&= \left|\frac{1}{2\pi i}\int_0^\infty \left[\frac{e^{(-x+i)\pi M}}{-x+i} - \frac{e^{(-x-i)\pi M}}{-x-i}\right]dx\right| \\
&= \left|\frac{1}{2\pi i}\right|\left|\int_0^\infty \left[\frac{e^{(-x+i)\pi M}}{-x+i} - \frac{e^{(-x-i)\pi M}}{-x-i}\right]dx\right| \\
&\le \frac{1}{2\pi}\int_0^\infty \left|\frac{e^{-x\pi M}e^{i\pi M}}{x-i} - \frac{e^{-x\pi M}e^{-i\pi M}}{x+i}\right|dx \\
&\le \frac{1}{2\pi}\int_0^\infty \left[\frac{e^{-x\pi M}}{|i|} + \frac{e^{-t\pi M}}{|i|}\right]dx \\
&= \frac{1}{2\pi}\int_0^\infty 2e^{-x\pi M}dx = \frac{1}{\pi^2 M} \to 0.
\end{aligned}$$

Hence we have

$$\int_0^\infty \operatorname{sinc}(t)dt = \frac{1}{2}.$$

By Lemma 1.21 we have

$$\int_{-\infty}^0 \operatorname{sinc}(t)dt = \int_\infty^0 \operatorname{sinc}(-t)(-dt) = \int_0^\infty \operatorname{sinc}(-t)dt = \int_0^\infty \operatorname{sinc}(t)dt = \frac{1}{2}.$$

This shows the claim. □

1.2 Fourier transform

In order to make certain singular positive real functions integrable over the real axis, we have to extend the definition of the Riemann-integral consisting improper integrals of the first and the second kind only. For further knowledge on other kinds of improper integrals see [6] chapter 15.4.

Definition 1.27. Let $x \in X$. The function x belongs to set A if there are finite sets $\hat{R} \subset \mathbb{R}$ and $\check{\hat{R}} \subset \mathbb{R}$ and functions $\hat{x} \in X$ and $\check{\hat{x}} \in X$ satisfying

$$\begin{aligned}
\hat{x}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^\infty x(t)e^{-i\omega t}dt \quad \forall \omega \in \mathbb{R}\setminus\hat{R} \\
\check{\hat{x}}(t) &= \int_{-\infty}^\infty \hat{x}(\omega)e^{i\omega t}d\omega \quad \forall t \in \mathbb{R}\setminus\check{\hat{R}}.
\end{aligned}$$

Definition 1.28. Assume that $x \in A$. We define

$$\begin{aligned}
\hat{R}_0 &= \{\omega \in \mathbb{R} \mid \int_{-\infty}^\infty x(t)e^{-i\omega t}dt \notin \mathbb{C}\} \\
\hat{R}_1 &= \{\omega \in \hat{R}_0 \mid \int_0^\infty (x(t)e^{-i\omega t} + x(-t)e^{i\omega t})dt \notin \mathbb{C}\}
\end{aligned}$$

and

$$\begin{aligned}
\hat{x}_0(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt, \quad \omega \in \mathbb{R}\setminus \hat{R}_0 \\
\hat{x}_1(\omega) &= \frac{1}{2\pi}\int_{0}^{\infty} (x(t)e^{-i\omega t} + x(-t)e^{i\omega t})dt, \quad \omega \in \hat{R}_0 \setminus \hat{R}_1.
\end{aligned}$$

Definition 1.29. *(Fourier transform)* We define

$$\mathscr{F} : A \to X, \quad \mathscr{F}x(\omega) = \begin{cases} \hat{x}_0(\omega), & \omega \in \mathbb{R}\setminus \hat{R}_0 \\ \hat{x}_1(\omega), & \omega \in \hat{R}_0 \setminus \hat{R}_1 \\ 0, & \omega \in \hat{R}_1. \end{cases}$$

The operator $\mathscr{F}$ is called Fourier transform and the function $\mathscr{F}x$ the Fourier transform of x.

Theorem 1.30. *We have the equation*

$$\mathscr{F}x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$$

for every $\omega \in \mathbb{R}$ whenever the right hand side exists for every $\omega \in \mathbb{R}$.
Proof: The integral expression converges everywhere. Hence $x \in A$ and $\hat{R}_0 = \mathbb{R}$. Furthermore

$$\mathscr{F}x(\omega) = \hat{x}_0(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt.$$

This shows the claim. □

Fourier transform has significance in some important applications of ordinary differential equations. The equation

$$\mathscr{F}x'(\omega) = i\omega\mathscr{F}(\omega) \tag{1.4}$$

is introduced in [1]. This is of special practical interest because many physical phenomena can be modeled as differential equations. For example the total charge of a condensator plate in an oscillator circuit satisfies a differential equation. Another interesting result concerning Fourier transform is

$$\mathscr{F}\tau_\alpha x(\omega) = e^{-i\alpha\omega}\mathscr{F}x(\omega), \tag{1.5}$$

that is also a consequence of results introduced in [1]. In order to introduce the use of translation we also show the result (1.5).

Lemma 1.31. *Assume $x \in A$ and*

$$\mathscr{F}x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt.$$

Then we have the equation

$$\mathscr{F}\tau_\alpha x(\omega) = e^{-i\omega\alpha}\mathscr{F}x(\omega)$$

Proof: We calculate

$$\begin{aligned}
e^{-i\omega\alpha}\mathscr{F}x(\omega) &= e^{-i\omega\alpha}\frac{1}{2\pi}\int_{-\infty}^{\infty}x(t)e^{-i\omega t}dt = \frac{1}{2\pi}e^{-i\omega\alpha}\int_{-\infty}^{\infty}x(t)e^{-i\omega t}dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega\alpha}x(t)e^{-i\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}x(t)e^{-i\omega t}e^{-i\omega\alpha}dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}x(t)e^{-i\omega t-i\omega\alpha}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}x(t)e^{-i\omega(t+\alpha)}dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}x(t-\alpha)e^{-i\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tau_\alpha x(t)e^{-i\omega t}dt \\
&= \mathscr{F}\tau_\alpha x(\omega)
\end{aligned}$$

to obtain the claim. □

An analog transmission line, that is, any medium between transmitter and receiver, with transmitter output signal that satisfies assumptions of Lemma 1.31 is an application of the equation (1.5).

1.3 Inverse Fourier transform

In this section we formulate and prove an important mathematical convergence theorem, Fourier inversion identity. It not only guarantees the existence of an inverse operator for Fourier transform but also gives an integral operator expression for the inverse transform.

Lemma 1.32. *We have the estimate*

$$\left|\int_x^{\infty}\operatorname{sinc}(t)dt\right| \leq \frac{C}{1+x}$$

for every $x \geq 0$.

Proof: We calculate and obtain

$$\begin{aligned}
\left|\int_x^{\infty}\operatorname{sinc}(t)dt\right| &= \left|\int_x^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\omega t}d\omega dt\right| = \left|\frac{1}{2\pi}\int_x^{\infty}\int_{-\pi}^{\pi}e^{i\omega t}d\omega dt\right| \\
&= \left|\frac{1}{2\pi}\right|\left|\int_x^{\infty}\int_{-\pi}^{\pi}e^{i\omega t}d\omega dt\right| = \frac{1}{2\pi}\left|\int_x^{\infty}\Big/_{-\pi}^{\pi}\frac{e^{i\omega t}}{it}dt\right| \\
&= \frac{1}{2\pi}\left|\int_x^{\infty}\frac{e^{i\pi t}-e^{-i\pi t}}{it}dt\right| \\
&= \frac{1}{2\pi}\left|\Big/_x^{\infty}\frac{1}{it}\left[\frac{e^{i\pi t}}{i\pi}-\frac{e^{-i\pi t}}{-i\pi}\right]\right. \\
&\qquad \left.-\int_x^{\infty}-\frac{1}{it^2}\left[\frac{e^{i\pi t}}{i\pi}-\frac{e^{-i\pi t}}{-i\pi}\right]dt\right| \\
&= \frac{1}{2\pi}\left|\Big/_x^{\infty}-\frac{e^{i\pi t}+e^{-i\pi t}}{\pi t}+\int_x^{\infty}\frac{e^{i\pi t}+e^{-i\pi t}}{\pi t^2}dt\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2\pi}\left[\left|\Big/_{x}^{\infty} -\frac{e^{i\pi t}+e^{-i\pi t}}{\pi t}\right| + \left|\int_x^{\infty} \frac{e^{i\pi t}+e^{-i\pi t}}{\pi t^2} dt\right|\right] \\
&\leq \frac{1}{2\pi}\left|\Big/_{x}^{\infty} -\frac{e^{i\pi t}+e^{-i\pi t}}{\pi t}\right| + \frac{1}{2\pi}\left|\int_x^{\infty} \frac{e^{i\pi t}+e^{-i\pi t}}{\pi t^2} dt\right| \\
&\leq \frac{1}{2\pi}\left|\frac{e^{i\pi x}+e^{-i\pi x}}{\pi x}\right| + \frac{1}{2\pi}\int_x^{\infty} \left|\frac{e^{i\pi t}+e^{-i\pi t}}{\pi t^2}\right| dt \\
&\leq \frac{1}{2\pi}\frac{2}{\pi x} + \frac{1}{2\pi}\int_x^{\infty} \frac{2}{\pi t^2} dt \leq \frac{1}{\pi^2 x} + \frac{1}{\pi^2 x} = \frac{2}{\pi^2}\frac{1}{x}
\end{aligned} \tag{1.6}
$$

for every $x > 0$. Integral of sinc from x to ∞ converges. Hence we have

$$
\lim_{x\to\infty} \int_x^{\infty} \operatorname{sinc}(t)dt = 0.
$$

This implies

$$
\left|\int_x^{\infty} \operatorname{sinc}(t)dt\right| \leq D. \tag{1.7}
$$

Combining estimates (1.6) and (1.7) we obtain

$$
\begin{aligned}
\left|\left(\frac{1}{D} + \frac{1}{\frac{2}{\pi^2}\frac{1}{x}}\right)\int_x^{\infty} \operatorname{sinc}(t)dt\right| &\leq \frac{1}{D}\left|\int_x^{\infty} \operatorname{sinc}(t)dt\right| + \frac{1}{\frac{2}{\pi^2}\frac{1}{x}}\left|\int_x^{\infty} \operatorname{sinc}(t)dt\right| \\
&\leq \frac{1}{D}D + \frac{1}{\frac{2}{\pi^2}\frac{1}{x}}\frac{2}{\pi^2}\frac{1}{x} = 2.
\end{aligned}
$$

Dividing both sides by $\frac{1}{D} + \frac{1}{\frac{2}{\pi^2}\frac{1}{x}}$ we obtain

$$
\left|\int_x^{\infty} \operatorname{sinc}(t)\right| \leq \frac{2}{\frac{1}{D} + \frac{1}{\frac{2}{\pi^2}\frac{1}{x}}} = \frac{4D}{2 + D\pi^2 x} \leq \frac{C}{1+x}, \quad \forall x > 0
$$

where $C = \frac{4D}{\min\{2, D\pi^2\}}$. Continuity of

$$
\left|\int_x^{\infty} \operatorname{sinc}(t)dt\right|
$$

now implies the claim. □

Definition 1.33. *We define substitution*

$$
\Big/_{-\infty}^{\infty} f(t) \;=\; \Big/_{-\infty}^{0} f(t) + \Big/_{0}^{\infty} f(t).
$$

Lemma 1.34. *We have the equation*

$$
\Big/_{-\infty}^{\infty} f(t) = \lim_{M\to\infty} f(M) - \lim_{M\to-\infty} f(M).
$$

Proof: We calculate

$$
\Big/_{-\infty}^{\infty} f(t) \;=\; \Big/_{-\infty}^{0} f(t) + \Big/_{0}^{\infty} f(t) = \lim_{M\to-\infty} \Big/_{M}^{0} f(t) + \lim_{M\to\infty} \Big/_{0}^{M} f(t)
$$

$$
\begin{aligned}
&= \lim_{M\to-\infty}(f(0)-f(M)) + \lim_{M\to\infty}(f(M)-f(0)) \\
&= \lim_{M\to-\infty} f(0) - \lim_{M\to-\infty} f(M) + \lim_{M\to\infty} f(M) - \lim_{M\to-\infty} f(0) \\
&= \lim_{M\to-\infty} f(0) - \lim_{M\to-\infty} f(0) - \lim_{M\to-\infty} f(M) + \lim_{M\to\infty} f(M) \\
&= \lim_{M\to\infty} f(M) - \lim_{M\to-\infty} f(M)
\end{aligned}
$$

Lemma 1.35. *Assume differentiable functions $f \in X$ and $g \in X$ such that complex numbers*

$$
\begin{aligned}
a_1 &= /_{-\infty}^{\infty} f(t)g(t) \\
a_2 &= \int_{-\infty}^{\infty} f(t)g'(t)dt
\end{aligned}
$$

exist. Then we have the equation

$$
\int_{-\infty}^{\infty} f'(t)g(t)dt = /_{-\infty}^{\infty} f(t)g(t) - \int_{-\infty}^{\infty} f(t)g'(t)dt.
$$

Proof: Assumptions imply the existence of the right hand side. We calculate and obtain

$$
\begin{aligned}
& /_{-\infty}^{\infty} f(t)g(t) - \int_{-\infty}^{\infty} f(t)g'(t)dt \\
&= /_{-\infty}^{0} f(t)g(t) + /_{0}^{\infty} f(t)g(t) - \left[\int_{-\infty}^{0} f(t)g'(t)dt + \int_{0}^{\infty} f(t)g'(t)dt\right] \\
&= /_{-\infty}^{0} f(t)g(t) + /_{0}^{\infty} f(t)g(t) - \int_{-\infty}^{0} f(t)g'(t)dt - \int_{0}^{\infty} f(t)g'(t)dt \\
&= /_{-\infty}^{0} f(t)g(t) - \int_{-\infty}^{0} f(t)g'(t)dt + /_{0}^{\infty} f(t)g(t) - \int_{0}^{\infty} f(t)g'(t)dt \\
&= \int_{-\infty}^{0} f'(t)g(t)dt + \int_{0}^{\infty} f'(t)g(t)dt = \int_{-\infty}^{\infty} f'(t)g(t)dt.
\end{aligned}
$$

This shows that the left hand side exists and equals to the right hand side. □

Lemma 1.36. *Assume that $x \in A$ and there is a function $\check{\hat{x}} \in X$ such that*

$$
\begin{aligned}
&\mathbf{1)} \qquad \check{\hat{x}}(t) = \int_{-\infty}^{\infty} \mathscr{F}x(\omega)e^{i\omega t}d\omega \\
&\mathbf{2)} \qquad \lim_{t\to\pm\infty} x(t) = 0 \\
&\mathbf{3)} \qquad \int_{-\infty}^{\infty} |x'(t)|\, dt \in \mathbb{R}
\end{aligned}
$$

where x' is the classical derivative of x. Then we have the equation

$$
\int_{-\infty}^{\infty} x(a-t)\mathrm{sinc}(\alpha t)\alpha dt = \int_{-\infty}^{\infty} x'(a-t)\int_0^t \mathrm{sinc}(\alpha\tau)\alpha d\tau dt,
$$

for every $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}$.
Proof: Fix $a \in \mathbb{R}$. We define

$$\begin{aligned} f(t) &= \int_0^t \operatorname{sinc}(\alpha\tau)\,\alpha d\tau \\ g(t) &= x(a-t). \end{aligned}$$

The function f is differentiable by Theorem 9.3.3 in [5]. The function g is differentiable as a composite of differentiable functions.

Sine cardinal is integrable over real axis by Lemma (1.26). Hence

$$\int_0^t \operatorname{sinc}(\alpha\tau)\,\alpha d\tau = \int_0^{\alpha t} \operatorname{sinc}(\tau) d\tau$$

is uniformly bounded over α. We apply assumption **2)** and calculate

$$\begin{aligned} 0 &= 0 \cdot \lim_{M\to\infty} \int_0^{\alpha M} \operatorname{sinc}(\tau)d\tau - 0 \cdot \lim_{M\to-\infty} \int_0^{\alpha M} \operatorname{sinc}(\tau)d\tau \\ &= \lim_{M\to\infty} x(a-M) \lim_{M\to\infty} \int_0^M \operatorname{sinc}(\alpha\tau)\alpha d\tau \\ &\quad - \lim_{M\to-\infty} x(a-M) \lim_{M\to-\infty} \int_0^M \operatorname{sinc}(\alpha\tau)\alpha d\tau \\ &= \lim_{M\to\infty} x(a-M) \int_0^M \operatorname{sinc}(\alpha\tau)\alpha d\tau \\ &\quad - \lim_{M\to-\infty} x(a-M) \int_0^M \operatorname{sinc}(\alpha\tau)\alpha d\tau \\ &= \lim_{M\to\infty} g(M)f(M) - \lim_{M\to-\infty} g(M)f(M) \\ &= \lim_{M\to\infty} f(M)g(M) - \lim_{M\to-\infty} f(M)g(M) \\ &= \Big/_{-\infty}^{\infty} f(t)g(t) \end{aligned}$$

This shows that the number a_1 in Lemma 1.35 exists. We apply assumption **3)** and estimate

$$\begin{aligned} \int_{-\infty}^{\infty} |f(t)g'(t)|dt &= \int_{-\infty}^{\infty} \left| \int_0^t \operatorname{sinc}(\alpha\tau)\alpha d\tau x'(a-t) \right| dt \\ &= \int_{-\infty}^{\infty} \left| \int_0^t \operatorname{sinc}(\alpha\tau)\alpha d\tau \right| \left| x'(a-t) \right| dt \\ &\leq \int_{-\infty}^{\infty} C|x'(a-t)|dt \\ &= C \int_{-\infty}^{\infty} |x'(a-t)|dt \\ &= C \int_{-\infty}^{\infty} |x'(t)|dt. \end{aligned}$$

This shows that the limit a_2 in Lemma 1.35 exists. Hence we have

$$\begin{aligned}
\int_{-\infty}^{\infty} x(a-t)\,\mathrm{sinc}(\alpha t)\alpha dt &= \int_{-\infty}^{\infty} g(t)f'(t)dt = \int_{-\infty}^{\infty} f'(t)g(t)dt \\
&= \Big/_{-\infty}^{\infty} f(t)g(t) - \int_{-\infty}^{\infty} f(t)g'(t)dt \\
&= 0 - \int_{-\infty}^{\infty} g'(t)f(t)dt \\
&= -\int_{-\infty}^{\infty} -x'(a-t)\int_0^t \mathrm{sinc}(\alpha\tau)\alpha d\tau dt \\
&= \int_{-\infty}^{\infty} x'(a-t)\int_0^t \mathrm{sinc}(\alpha\tau)\alpha d\tau dt.
\end{aligned}$$

This shows the claim. □

Lemma 1.37. *Assume that $x \in A$ and there is a function $\check{\tilde{x}} \in X$ such that*

$$\textbf{1)} \qquad \check{\tilde{x}}(t) = \int_{-\infty}^{\infty} \mathscr{F}x(\omega)e^{i\omega t}d\omega$$

$$\textbf{2)} \qquad \lim_{t\to\pm\infty} x(t) = 0$$

$$\textbf{3)} \qquad \int_{-\infty}^{\infty} |x'(t)|\,dt \in \mathbb{R}$$

where x' is the classical derivative of x. Then we have the equation

$$\int_{-M}^{M}\int_{-\infty}^{\infty} x(\tau)e^{i\omega(t-\tau)}d\tau\,d\omega = \int_{-\infty}^{\infty}\int_{-M}^{M} x(\tau)e^{i\omega(t-\tau)}d\omega\,d\tau$$

for every $M \in \mathbb{R}$.
Proof: Let $\omega \in [-M,-a] \cup [a,M]$, where $M > a > 0$. Assume that both M_1 and M_2 go to infinity or both M_1 and M_2 go to minus infinity. We calculate and obtain a uniform estimate

$$\begin{aligned}
\left|\int_{M_1}^{M_2} x(\tau)e^{-i\omega\tau}\,d\tau\right| &= \left|\Big/_{M_1}^{M_2} x(\tau)\int_0^{\tau} e^{-i\omega\tau'}d\tau' - \int_{M_1}^{M_2} x'(\tau)\int_0^{\tau} e^{-i\omega\tau'}d\tau' d\tau\right| \\
&= \left|\Big/_{M_1}^{M_2} x(\tau)\frac{1}{-i\omega}\left(e^{-i\omega\tau}-1\right) - \int_{M_1}^{M_2} x'(\tau)\frac{1}{-i\omega}\left(e^{-i\omega\tau}-1\right)d\tau\right| \\
&= \left|\Big/_{M_1}^{M_2} x(\tau)\frac{1}{-i\omega}e^{-i\omega\tau} - \int_{M_1}^{M_2} x'(\tau)\frac{1}{-i\omega}e^{-i\omega\tau}\,d\tau\right| \\
&\le \left|\frac{1}{-i\omega}x(M_2)e^{-i\omega M_2} - \frac{1}{-i\omega}x(M_1)e^{-i\omega M_1}\right| + \left|\frac{1}{-i\omega}\int_{M_1}^{M_2} x'(\tau)e^{-i\omega\tau}\,d\tau\right| \\
&\le \frac{1}{a}|x(M_2)| + \frac{1}{a}|x(M_1)| + \frac{1}{a}\int_{M_1}^{M_2} |x'(\tau)|\,d\tau \to 0.
\end{aligned}$$

We apply Lemma 1.22 (Cauchy uniform convergence criterion for integrals) to obtain uniform convergence on intervals $[-M,-a]$ and $[a,M]$. We interchange the order of limit and integration by Lemma 1.23 and obtain

$$\begin{aligned}\int_a^M \int_0^\infty x(\tau)e^{-i\omega\tau}d\tau e^{i\omega t}d\omega &= \int_a^M \lim_{M_1\to\infty}\int_0^{M_1} x(\tau)e^{-i\omega\tau}d\tau e^{i\omega t}d\omega \\ &= \lim_{M_1\to\infty}\int_a^M \int_0^{M_1} x(\tau)e^{-i\omega\tau}d\tau e^{i\omega t}d\omega \\ &= \lim_{M_1\to\infty}\int_0^{M_1} \int_a^M x(\tau)e^{-i\omega\tau}e^{i\omega t}d\omega d\tau \\ &= \int_0^\infty \int_a^M x(\tau)e^{-i\omega\tau}e^{i\omega t}d\omega d\tau.\end{aligned}$$

Another similar calculation shows

$$\int_a^M \int_{-\infty}^0 x(\tau)e^{-i\omega\tau}d\tau e^{i\omega t}d\omega = \int_{-\infty}^0 \int_a^M x(\tau)e^{-i\omega\tau}e^{i\omega t}d\omega d\tau.$$

Hence we have

$$\begin{aligned}&\int_{-\infty}^0 \int_a^M x(\tau)e^{-i\omega\tau}e^{i\omega t}d\omega d\tau + \int_0^\infty \int_a^M x(\tau)e^{-i\omega\tau}d\tau e^{i\omega t}d\omega d\tau \\ &= \int_a^M \int_{-\infty}^0 x(\tau)e^{-i\omega\tau}d\tau e^{i\omega t}d\omega + \int_a^M \int_0^\infty x(\tau)e^{-i\omega\tau}d\tau e^{i\omega t}d\omega \\ &= \int_a^M \left[\int_{-\infty}^0 x(\tau)e^{-i\omega\tau}d\tau + \int_0^\infty x(\tau)e^{-i\omega\tau}d\tau\right]e^{i\omega t}d\omega.\end{aligned}$$

By definition of improper Riemann-integral over real axis we have

$$\int_{-\infty}^\infty \int_a^M x(\tau)e^{-i\omega\tau}e^{i\omega t}d\omega d\tau = \int_a^M \int_{-\infty}^\infty x(\tau)e^{-i\omega\tau}d\tau e^{i\omega t}d\omega$$

for every $a > 0$ and $M > 0$ whenever $a < M$. Another similar reasoning shows

$$\int_{-\infty}^\infty \int_{-M}^{-a} x(\tau)e^{i\omega(t-\tau)}d\omega d\tau = \int_{-M}^{-a} \int_{-\infty}^\infty x(\tau)e^{-i\omega\tau}d\tau e^{i\omega t}\, d\omega$$

for every $a > 0$ and $M > 0$ whenever $a < M$. Hence we obtain

$$\begin{aligned}&\int_{-M}^M \int_{-\infty}^\infty x(\tau)e^{i\omega(t-\tau)}d\tau d\omega - \int_{-\pi a}^{\pi a} \int_{-\infty}^\infty x(\tau)e^{i\omega(t-\tau)}d\tau d\omega \\ &= \int_{-M}^{-\pi a} \int_{-\infty}^\infty x(\tau)e^{i\omega(t-\tau)}d\tau d\omega + \int_{\pi a}^M \int_{-\infty}^\infty x(\tau)e^{i\omega(t-\tau)}d\tau d\omega \\ &= \int_{-\infty}^\infty \int_{-M}^{-\pi a} x(\tau)e^{i\omega(t-\tau)}d\tau d\omega + \int_{-\infty}^\infty \int_{\pi a}^M x(\tau)e^{i\omega(t-\tau)}d\tau d\omega \\ &= \int_{-\infty}^\infty \left[\int_{-M}^{-\pi a} x(\tau)e^{i\omega(t-\tau)}d\tau + \int_{\pi a}^M x(\tau)e^{i\omega(t-\tau)}d\tau\right]d\omega \\ &= \int_{-\infty}^\infty \left[\int_{-M}^M x(\tau)e^{i\omega(t-\tau)}d\tau - \int_{-\pi a}^{\pi a} x(\tau)e^{i\omega(t-\tau)}d\tau\right]d\omega \\ &= \int_{-\infty}^\infty \int_{-M}^M x(\tau)e^{i\omega(t-\tau)}d\omega d\tau - \int_{-\infty}^\infty \int_{-\pi a}^{\pi a} x(\tau)e^{i\omega(t-\tau)}d\omega d\tau. \qquad (1.8)\end{aligned}$$

For the last term I_2 on the last line of (1.8) we calculate

$$\begin{aligned}
|I_2| &= \left|\int_{-\infty}^{\infty}\int_{-\pi a}^{\pi a} x(\tau)e^{i\omega(t-\tau)}d\omega\, d\tau\right| = \left|\int_{-\infty}^{\infty} x(\tau)\int_{-\pi a}^{\pi a} e^{i\omega(t-\tau)}d\omega\, d\tau\right| \\
&= \left|\int_{-\infty}^{\infty} x(t-\tau)\int_{-\pi a}^{\pi a} e^{i\omega\tau}d\omega\, d\tau\right| = \left|\int_{-\infty}^{\infty} x(t-\tau)\int_{-\pi}^{\pi} e^{i\omega a\tau}d\omega\, a\, d\tau\right|.
\end{aligned}$$

We define

$$I_4 = \int_{-\infty}^{\infty} x(t-\tau)\,\mathrm{sinc}(a\tau)\, a\, d\tau$$

and calculate

$$\begin{aligned}
|I_4| &= \left|\int_{-\infty}^{\infty} x(t-\tau)\,\mathrm{sinc}(a\tau)\, a\, d\tau\right| = \left|\int_{-\infty}^{\infty} x'(t-\tau)\int_0^{\tau}\mathrm{sinc}(a\tau')\, a d\tau' d\tau\right| \\
&= \left|\int_{-\infty}^{\infty} x'(t-\tau)\int_0^{a\tau}\mathrm{sinc}(\tau'')\, d\tau'' d\tau\right| \\
&= \left|\int_{-\infty}^{-\frac{1}{\sqrt{a}}} x'(t-\tau)\int_0^{a\tau}\mathrm{sinc}(\tau'')d\tau'' d\tau\right. \\
&\quad + \int_{-\frac{1}{\sqrt{a}}}^{+\frac{1}{\sqrt{a}}} x'(t-\tau)\int_0^{a\tau}\mathrm{sinc}(\tau'')d\tau'' d\tau \\
&\quad \left. + \int_{+\frac{1}{\sqrt{a}}}^{\infty} x'(t-\tau)\int_0^{a\tau}\mathrm{sinc}(\tau'')d\tau'' d\tau\right| \\
&= \left|\int_{-\infty}^{-\frac{1}{\sqrt{a}}} x'(t-\tau)\int_0^{a\tau}\mathrm{sinc}(\tau')\, d\tau' d\tau\right. \\
&\quad + \Big/_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}} - x(t-\tau)\int_0^{a\tau}\mathrm{sinc}(\tau')d\tau' \\
&\quad + \int_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}} x(t-\tau)\,\mathrm{sinc}(a\tau)ad\tau \\
&\quad \left. + \int_{\frac{1}{\sqrt{a}}}^{\infty} x'(t-\tau)\int_0^{a\tau}\mathrm{sinc}(\tau')\, d\tau' d\tau\right| \\
&\le C\int_{-\infty}^{-\frac{1}{\sqrt{a}}} |x'(t-\tau)|\, d\tau + \left|x(t-\frac{1}{\sqrt{a}})\int_0^{\sqrt{a}}\mathrm{sinc}(\tau')d\tau'\right| \\
&\quad + \left|x(t+\frac{1}{\sqrt{a}})\int_0^{-\sqrt{a}}\mathrm{sinc}(\tau')d\tau'\right| + \left|\int_{-\sqrt{a}}^{\sqrt{a}} x(\frac{t-\tau}{a})\,\mathrm{sinc}(\tau)d\tau\right| \\
&\quad + C\int_{\frac{1}{\sqrt{a}}}^{\infty} |x'(t-\tau)|\, d\tau \\
&\le C\int_{-\infty}^{-\frac{1}{\sqrt{a}}} |x'(\tau)|\, d\tau + 4\sqrt{a}||x||_\infty + C\int_{\frac{1}{\sqrt{a}}}^{\infty} |x'(\tau)|\, d\tau \\
&\to 0
\end{aligned}$$

for every $t \in \mathbb{R}$ as $a \to 0$. Hence we have

$$\begin{aligned}
I_2 &= 2\pi\frac{1}{2\pi}I_2 = 2\pi\frac{1}{2\pi}\int_{-\infty}^{\infty} x(t-\tau)\int_{-\pi}^{\pi} e^{i\omega a\tau}\,d\omega\, a\, d\tau \\
&= 2\pi\int_{-\infty}^{\infty}\frac{1}{2\pi}x(t-\tau)\int_{-\pi}^{\pi} e^{i\omega a\tau}\,d\omega\, a\, d\tau \\
&= 2\pi\int_{-\infty}^{\infty} x(t-\tau)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega a\tau}\,d\omega\, a\, d\tau \\
&= 2\pi\int_{-\infty}^{\infty} x(t-\tau)\mathrm{sinc}(a\tau)a\, d\tau = 2\pi I_4 \to 0
\end{aligned}$$

as $a \to 0$. Transform $\mathscr{F}x$ has improper Riemann-integral on the interval $[-a, a]$ by assumption. Assume that $a \to 0$. We define

$$I_3 = \int_{-\pi a}^{\pi a} \mathscr{F}x(\omega)e^{i\omega t}\,d\omega.$$

Hence we have $I_3 \to 0$ as $a \to 0$. We calculate and obtain

$$\begin{aligned}
I_1 &= 2\pi\frac{1}{2\pi}I_1 = 2\pi\frac{1}{2\pi}\int_{-\pi a}^{\pi a}\int_{-\infty}^{\infty} x(\tau)e^{i\omega(t-\tau)}\,d\tau d\omega \\
&= 2\pi\int_{-\pi a}^{\pi a}\frac{1}{2\pi}\int_{-\infty}^{\infty} x(\tau)e^{-i\omega\tau}\,d\tau e^{i\omega t}\,d\omega \\
&= 2\pi\int_{-\pi a}^{\pi a}\mathscr{F}x(\omega)e^{i\omega t}\,d\omega = 2\pi I_3 \to 0
\end{aligned}$$

as $a \to 0$. This shows that the last term I_1 on the first line of (1.8) approaches to zero as a approaches to zero. This completes the proof. □

Lemma 1.38. *Assume that $x \in A$ and there is a function $\check{x} \in X$ such that*

$$\textbf{1)} \qquad \check{x}(t) = \int_{-\infty}^{\infty} \mathscr{F}x(\omega)e^{i\omega t}\,d\omega$$

$$\textbf{2)} \qquad \lim_{t\to\pm\infty} x(t) = 0$$

$$\textbf{3)} \qquad \int_{-\infty}^{\infty} |x'(t)|\,dt \in \mathbb{R}$$

where x' is the classical derivative of x. Assume that $M \to \infty$. Then we have limits

$$\int_{0}^{\infty} x'(t-\tau)\int_{0}^{M\tau}\mathrm{sinc}(s)\,ds\,d\tau \;\to\; \frac{1}{2}x(t) \tag{1.9}$$

$$\int_{-\infty}^{0} x'(t-\tau)\int_{0}^{M\tau}\mathrm{sinc}(s)\,ds\,d\tau \;\to\; \frac{1}{2}x(t). \tag{1.10}$$

Proof: We apply assumption **2**) to obtain results

$$\begin{aligned}
\int_{0}^{M} x'(t-\tau)d\tau &= -\int_{t}^{t-M} x'(\tau)d\tau = -x(t-M)+x(t) \;\to\; x(t) \\
\int_{-M}^{0} x'(t-\tau)d\tau &= -\int_{t+M}^{t} x'(\tau)d\tau = -x(t)+x(t+M) \;\to\; -x(t),
\end{aligned}$$

when $M \to \infty$. Hence we have

$$\begin{aligned}\int_0^\infty x'(t-\tau)d\tau &= x(t)\\ \int_{-\infty}^0 x'(t-\tau)d\tau &= -x(t).\end{aligned}$$

Assume that $M \to \infty$. We apply assumption **3)** and calculate

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{|x'(t-\tau)|}{1+|M\tau|}d\tau &= \int_{-\infty}^{-\frac{1}{\sqrt{M}}} \frac{|x'(t-\tau)|}{1+|M\tau|}d\tau + \int_{-\frac{1}{\sqrt{M}}}^{\frac{1}{\sqrt{M}}} \frac{|x'(t-\tau)|}{1+|M\tau|}d\tau\\ &\quad + \int_{\frac{1}{\sqrt{M}}}^{\infty} \frac{|x'(t-\tau)|}{1+|M\tau|}d\tau\\ &\leq \frac{1}{\sqrt{M}}\int_{-\infty}^{\infty} |x'(t-\tau)|\,d\tau + \int_{-\frac{1}{\sqrt{M}}}^{\frac{1}{\sqrt{M}}} |x'(t-\tau)|d\tau\\ &\to 0\end{aligned}$$

to obtain an estimate that allows to analyze Fourier convergence. Assume again that $M \to \infty$. We define

$$\begin{aligned}I_1 &= \int_0^\infty x'(t-\tau)\int_0^{M\tau} \mathrm{sinc}(s)dsd\tau - \frac{1}{2}x(t)\\ I_2 &= \int_{-\infty}^0 x'(t-\tau)\int_0^{M\tau} \mathrm{sinc}(s)dsd\tau - \frac{1}{2}x(t)\end{aligned}$$

and estimate

$$\begin{aligned}|I_1| &= \left|\int_0^\infty x'(t-\tau)\int_0^{M\tau} \mathrm{sinc}(s)dsd\tau - \frac{1}{2}x(t)\right|\\ &= \left|\int_0^\infty x'(t-\tau)\int_0^{\infty} \mathrm{sinc}(s)dsd\tau - \frac{1}{2}x(t)\right.\\ &\quad \left.-\int_0^\infty x'(t-\tau)\int_{M\tau}^{\infty} \mathrm{sinc}(s)dsd\tau\right|\\ &= \left|\int_0^\infty \mathrm{sinc}(s)ds\int_0^{\infty} x'(t-\tau)d\tau - \frac{1}{2}x(t)\right.\\ &\quad \left.-\int_0^\infty x'(t-\tau)\int_{M\tau}^{\infty} \mathrm{sinc}(s)dsd\tau\right|\\ &= \left|\frac{1}{2}x(t) - \frac{1}{2}x(t)\right.\\ &\quad \left.-\int_0^\infty x'(t-\tau)\int_{M\tau}^{\infty} \mathrm{sinc}(s)dsd\tau\right|\\ &= \left|\int_0^\infty x'(t-\tau)\int_{M\tau}^{\infty} \mathrm{sinc}(s)dsd\tau\right|\\ &\leq \int_0^\infty \left|x'(t-\tau)\int_{M\tau}^{\infty} \mathrm{sinc}(s)ds\right|d\tau\\ &= \int_0^\infty \left|x'(t-\tau)\right|\left|\int_{M\tau}^{\infty} \mathrm{sinc}(s)ds\right|d\tau\end{aligned}$$

$$\leq \int_0^{\infty} |x'(t-\tau)| \frac{C}{1+|M\tau|} d\tau \leq C \int_{-\infty}^{\infty} \frac{|x'(t-\tau)|}{1+|M\tau|} d\tau \to 0$$

to obtain (1.9). We calculate further

$$\begin{aligned}
|I_2| &= \left| \int_{-\infty}^{0} x'(t-\tau) \int_0^{M\tau} \operatorname{sinc}(s) ds d\tau - \frac{1}{2}x(t) \right| \\
&= \left| \int_{-\infty}^{0} x'(t-\tau) \int_0^{-\infty} \operatorname{sinc}(s) ds d\tau - \frac{1}{2}x(t) \right. \\
&\quad \left. - \int_{-\infty}^{0} x'(t-\tau) \int_{M\tau}^{-\infty} \operatorname{sinc}(s) ds d\tau \right| \\
&= \left| \int_{-\infty}^{0} x'(t-\tau) \int_0^{\infty} \operatorname{sinc}(-s)(-ds) d\tau - \frac{1}{2}x(t) \right. \\
&\quad \left. - \int_{-\infty}^{0} x'(t-\tau) \int_{-M\tau}^{\infty} \operatorname{sinc}(-s)(-ds) d\tau \right| \\
&= \left| - \int_{-\infty}^{0} x'(t-\tau) \int_0^{\infty} \operatorname{sinc}(s) ds d\tau - \frac{1}{2}x(t) \right. \\
&\quad \left. + \int_{-\infty}^{0} x'(t-\tau) \int_{-M\tau}^{\infty} \operatorname{sinc}(s) ds d\tau \right| \\
&= \left| - \int_0^{\infty} \operatorname{sinc}(s) ds \int_{-\infty}^{0} x'(t-\tau) d\tau - \frac{1}{2}x(t) \right. \\
&\quad \left. + \int_{-\infty}^{0} x'(t-\tau) \int_{-M\tau}^{\infty} \operatorname{sinc}(s) ds d\tau \right| \\
&= \left| -\frac{1}{2}\Big[- x(t) \Big] - \frac{1}{2}x(t) \right. \\
&\quad \left. + \int_{-\infty}^{0} x'(t-\tau) \int_{-M\tau}^{\infty} \operatorname{sinc}(s) ds d\tau \right| \\
&= \left| \int_{-\infty}^{0} x'(t-\tau) \int_{-M\tau}^{\infty} \operatorname{sinc}(s) ds d\tau \right| \\
&\leq \int_{-\infty}^{0} \left| x'(t-\tau) \int_{-M\tau}^{\infty} \operatorname{sinc}(s) ds \right| d\tau \\
&= \int_{-\infty}^{0} \left| x'(t-\tau) \right| \left| \int_{-M\tau}^{\infty} \operatorname{sinc}(s) ds \right| d\tau \\
&\leq \int_{-\infty}^{0} |x'(t-\tau)| \frac{C}{1+|-M\tau|} d\tau \\
&\leq C \int_{-\infty}^{\infty} \frac{|x'(t-\tau)|}{1+|M\tau|} d\tau \to 0
\end{aligned}$$

to obtain (1.10). This completes the proof. □

Lemma 1.39. *Assume that $x \in A$ and there is a function $\check{x} \in X$ such that*

$$\textbf{1)} \qquad \check{x}(t) = \int_{-\infty}^{\infty} \mathscr{F}x(\omega)e^{i\omega t}d\omega$$

$$\textbf{2)} \qquad \lim_{t\to\pm\infty} x(t) = 0$$

$$\textbf{3)} \qquad \int_{-\infty}^{\infty} |x'(t)|\, dt \in \mathbb{R}$$

where x' is the classical derivative of x. Then we have the equation

$$\int_{-\pi M}^{\pi M} \mathscr{F}x(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} x'(t-\tau)\int_{0}^{M\tau} \operatorname{sinc}(s)\, ds d\tau.$$

Proof: Fix $t \in \mathbb{R}$. We calculate

$$\begin{aligned}
\int_{-\pi M}^{\pi M} \mathscr{F}x(\omega)e^{i\omega t}d\omega &= \int_{-\pi M}^{\pi M} \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\tau)e^{-i\omega\tau}d\tau e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi M}^{\pi M}\int_{-\infty}^{\infty} x(\tau)e^{-i\omega\tau}d\tau e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi M}^{\pi M} e^{i\omega t}\int_{-\infty}^{\infty} x(\tau)e^{-i\omega\tau}d\tau d\omega \\
&= \frac{1}{2\pi}\int_{-\pi M}^{\pi M}\int_{-\infty}^{\infty} e^{i\omega t}x(\tau)e^{-i\omega\tau}d\tau d\omega \\
&= \frac{1}{2\pi}\int_{-\pi M}^{\pi M}\int_{-\infty}^{\infty} x(\tau)e^{i\omega t}e^{-i\omega\tau}d\tau d\omega \\
&= \frac{1}{2\pi}\int_{-\pi M}^{\pi M}\int_{-\infty}^{\infty} x(\tau)e^{i\omega t-i\omega\tau}d\tau d\omega \\
&= \frac{1}{2\pi}\int_{-\pi M}^{\pi M}\int_{-\infty}^{\infty} x(\tau)e^{i\omega(t-\tau)}d\tau d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\pi M}^{\pi M} x(\tau)e^{i\omega(t-\tau)}d\omega d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\tau)\int_{-\pi M}^{\pi M} e^{i\omega(t-\tau)}d\omega d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t-\tau)\int_{-\pi M}^{\pi M} e^{i\omega\tau}d\omega d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t-\tau)\int_{-\pi}^{\pi} e^{i\omega M\tau}Md\omega d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t-\tau)\int_{-\pi}^{\pi} Me^{i\omega M\tau}d\omega d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t-\tau)M\int_{-\pi}^{\pi} e^{i\omega M\tau}d\omega d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t-\tau)\int_{-\pi}^{\pi} e^{i\omega M\tau}d\omega M d\tau
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} x(t-\tau) \int_{-\pi}^{\pi} e^{i\omega M\tau} d\omega M d\tau = \int_{-\infty}^{\infty} x(t-\tau) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega M\tau} d\omega M d\tau$$
$$= \int_{-\infty}^{\infty} x(t-\tau)\mathrm{sinc}(M\tau) M d\tau$$

and obtain a nice formula. We calculate further

$$\begin{aligned} \int_{-\pi M}^{\pi M} \mathscr{F}x(\omega)e^{i\omega t}d\omega &= \int_{-\infty}^{\infty} x(t-\tau)\mathrm{sinc}(M\tau)Md\tau \\ &= \int_{-\infty}^{\infty} x'(t-\tau)\int_0^{\tau} \mathrm{sinc}(Ms)Mdsd\tau \\ &= \int_{-\infty}^{\infty} x'(t-\tau)\int_0^{M\tau} \mathrm{sinc}(s)dsd\tau \end{aligned}$$

and obtain the claim. □

Theorem 1.40. *(Fourier inversion identity)*
Assume that $x \in A$ and there is a function $\check{\hat{x}} \in X$ such that

$$\textbf{1)} \qquad \check{\hat{x}}(t) = \int_{-\infty}^{\infty} \mathscr{F}x(\omega)e^{i\omega t}d\omega$$

$$\textbf{2)} \qquad \lim_{t\to\pm\infty} x(t) = 0$$

$$\textbf{3)} \qquad \int_{-\infty}^{\infty} |x'(t)|\, dt \in \mathbb{R}$$

where x' is the classical derivative of x. Then we have the convergence result

$$x(t) = \int_{-\infty}^{\infty} \mathscr{F}x(\omega)e^{i\omega t}d\omega.$$

for every $t \in \mathbb{R}$.
Proof: Fix $t \in \mathbb{R}$. Assume that $M \to \infty$. We calculate and apply lemmas 1.36, 1.37 and 1.38 to obtain

$$\begin{aligned} \left|\int_{-M}^{M} \mathscr{F}x(\omega)e^{i\omega t}d\omega - x(t)\right| &= \left|\int_{-\infty}^{\infty} x'(t-\tau)\int_0^{\frac{M}{\pi}\tau} \mathrm{sinc}(s)dsd\tau - x(t)\right| \\ &\leq \left|\int_{0}^{\infty} x'(t-\tau)\int_0^{\frac{M}{\pi}\tau} \mathrm{sinc}(s)dsd\tau - \frac{1}{2}x(t)\right| \\ &\quad + \left|\int_{-\infty}^{0} x'(t-\tau)\int_0^{\frac{M}{\pi}\tau} \mathrm{sinc}(s)dsd\tau - \frac{1}{2}x(t)\right| \\ &\to 0. \end{aligned}$$

Both left and right hand side of the claim exist by assumption. We calculate

$$\begin{aligned} \int_{-\infty}^{\infty} \mathscr{F}x(\omega)e^{i\omega t}d\omega &= \int_{-\infty}^{0} \mathscr{F}x(\omega)e^{i\omega t}d\omega + \int_{0}^{\infty} \mathscr{F}x(\omega)e^{i\omega t}d\omega \\ &= \lim_{M\to\infty}\int_{-M}^{0} \mathscr{F}x(\omega)e^{i\omega t}d\omega + \lim_{M\to\infty}\int_{0}^{M} \mathscr{F}x(\omega)e^{i\omega t}d\omega \end{aligned}$$

$$
\begin{aligned}
&= \lim_{M\to\infty}\left[\int_{-M}^{0}\mathscr{F}x(\omega)e^{i\omega t}d\omega + \int_{0}^{M}\mathscr{F}x(\omega)e^{i\omega t}d\omega\right] \\
&= \lim_{M\to\infty}\int_{-M}^{M}\mathscr{F}x(\omega)e^{i\omega t}d\omega = x(t).
\end{aligned}
$$

This shows the claim. □

Corollary 1.41. *Assume that function $x \in X$ satisfies assumptions of Theorem* 1.40. *Then Fourier inversion identity decomposes x to oscillations. Fourier transform is a linear map that identifies oscillations existing in x.*
Proof: The claim follow directly from Theorem 1.40. □

Corollary 1.41 suggests that linear partial differential equations can be solved so that every decomposed oscillation is solved separately and that the solution is obtained as integral of previously calculated separate fundamental solutions. This method is considered in [3].

For many functions $x \in X$ that are of interest in mathematical physics, there is $y \in X$ such that x has the integral representation

$$
x(t) = \int_{-\infty}^{\infty} y(\omega)e^{i\omega t}d\omega. \tag{1.11}
$$

Definition 1.42. An integral is said to be a Fourier integral if its form is equivalent to the form of the right hand side of (1.11).

Definition 1.43. Any representation of a function $x \in X$ is said to be a Fourier integral representation if it is of the form (1.11).

Fourier transform is useful because it gives an explicit formula to calculate a suitable y. The formula is

$$
y = \mathscr{F}(x).
$$

This was shown by Theorem 1.40.

Definition 1.44. Assume that $x \in A$. We define

$$
\begin{aligned}
\check{\hat{R}}_0 &= \{\omega \in \mathbb{R} \mid \int_{-\infty}^{\infty}\mathscr{F}x(\omega)e^{i\omega t}d\omega \notin \mathbb{C}\} \\
\check{\hat{R}}_1 &= \{\omega \in \hat{R}_0 \mid \int_{0}^{\infty}(\mathscr{F}x(\omega)e^{i\omega t} + \mathscr{F}x(-\omega)e^{-i\omega t})d\omega \notin \mathbb{C}\}
\end{aligned}
$$

and

$$
\begin{aligned}
\check{\hat{x}}_0(\omega) &= \int_{-\infty}^{\infty}\mathscr{F}x(\omega)e^{i\omega t}d\omega, \quad \omega \in \mathbb{R}\setminus\check{\hat{R}}_0 \\
\check{\hat{x}}_1(\omega) &= \int_{0}^{\infty}(\mathscr{F}x(\omega)e^{i\omega t} + \mathscr{F}x(-\omega)e^{-i\omega t})d\omega, \quad \omega \in \check{\hat{R}}_0\setminus\check{\hat{R}}_1.
\end{aligned}
$$

Definition 1.45. We define

$$
\mathscr{F}' : \mathscr{F}A \to X, \quad \mathscr{F}'x(t) = \begin{cases} \check{\hat{x}}_0(t), & t \in \mathbb{R}\setminus\check{\hat{R}}_0 \\ \check{\hat{x}}_1(t), & t \in \check{\hat{R}}_0\setminus\check{\hat{R}}_1 \\ 0, & t \in \check{\hat{R}}_1. \end{cases}
$$

Definition 1.46. *The set of all functions $x \in A$, that satisfy the assumptions of Theorem 1.40 is D.*

Theorem 1.47. *(Fourier inversion theorem)*
Fourier transform $\mathscr{F} : D \to \mathscr{F}(D)$ has left inverse

$$\mathscr{F}^{-1} : \mathscr{F}(D) \to D, \quad \mathscr{F}^{-1}x(t) = \mathscr{F}'x(t).$$

Proof: Fix $x \in D$. We calculate applying Theorem 1.40 to obtain

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} \mathscr{F}x(\omega)e^{i\omega t}d\omega = \mathscr{F}'\mathscr{F}x(t) \\ \mathscr{F}'\mathscr{F}x &= x = Ix \qquad (1.12) \\ \mathscr{F}'\mathscr{F} &= I. \qquad (1.13) \end{aligned}$$

The equation (1.12) shows that $\mathscr{F}'$ is defined on $\mathscr{F}(D)$. The equation (1.13) shows the left inverse axiom. Hence the claim follows. □

Definition 1.48. Fourier-domain is the domain of definition of Fourier transform of a function.

Definition 1.49. Fourier analysis is defined as analysis on Fourier-domain. This includes calculation of Fourier transform.

The result 1.40 exist also for functions of several variables. Fourier transform and inverse Fourier transform are given by

$$\mathscr{F}f(k) = \int_{\mathbb{R}^n} f(x)e^{-ik\cdot x}dx \qquad (1.14)$$

$$\mathscr{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(k)e^{ik\cdot x}dk. \qquad (1.15)$$

These transforms are introduced in [1].

In order to make calculations more simple, we will place the constant $\frac{1}{(2\pi)^n}$ in front of Fourier transform instead of inverse Fourier transform.

Definition 1.50. Assume that function $f : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$ is defined over space $\mathbb{R}^3$ and time $[0, \infty)$. Then we define

$$\mathscr{F}_x f(k,t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} f(x,t)e^{-ik\cdot x}dx$$

whenever the integral exists for every $k \in S$ for every $t \in [0, \infty)$.

Definition 1.51. Assume that function $f : S \times \mathbb{R} \to \mathbb{C}$ is defined over wave vector space S and time $\mathbb{R}$. Then we define

$$\mathscr{F}_t f(k,\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} f(k,t)e^{-i\omega t}dt$$

whenever the integral exists for every $k \in S$ for every $\omega \in \mathbb{R}$.

For functions $f \in W$, that satisfy sufficient assumptions, for fixed $k \in S$ equations (1.14) and (1.15) impliy

$$\mathscr{F}_t^{-1} f(k,t) = \int_{\mathbb{R}^3} f(k,\omega) e^{i\omega t} d\omega. \tag{1.16}$$

Hence (1.16) holds for every $k \in \mathbb{R}^3$.

For functions $f : S \times [0,\infty) \to \mathbb{C}$ for fixed $t \in [0,\infty)$ equations (1.14) and (1.15) imply

$$\mathscr{F}_x^{-1} f(x,t) = \int_{\mathbb{R}^3} f(k,t) e^{ik\cdot x} dk. \tag{1.17}$$

Hence (1.17) holds for every $t \in [0,\infty)$.

1.4 Fourier-domain representation of product

Definition 1.52. Assume $f \in X$ and $g \in X$. We define convolution

$$f * g(x) = \int_{-\infty}^{\infty} f(t) g(x-t) dt$$

whenever the integral exists for every $x \in \mathbb{R}$.

Definition 1.53. Assume $f : \mathbb{R}^n \to \mathbb{C}$ and $g : \mathbb{R}^n \to \mathbb{C}$. We define convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(t) g(x-t) dt$$

whenever the integral exists for every $x \in \mathbb{R}^n$.

Definition 1.54. Assume functions $f : S \times [0,\infty) \to \mathbb{C}$ and $g : S \times [0,\infty) \to \mathbb{C}$. We define convolution

$$f * g(k,t) = \int_{S\setminus\{k\}} f(k',t) g(k-k',t) dk'$$

whenever the integral exists for every $k \in S$ and $t \in [0,\infty)$.

Definition 1.55. If the domain of integration is bigger than the domain of definition of an integrand, all integrands are zero-extended such that all necessary algebraic operations are defined for every element of the domain of integration.

Definition 1.56. Assume $f \in W$ and $g \in W$. We define convolution

$$(f * g)(k,\omega) = \int_{\mathbb{R}^3} \int_{\mathbb{R}} f(k',\omega') g(k-k',\omega-\omega') d\omega' dk'$$

whenever the integral exists for every $k \in S$ and $\omega \in \mathbb{R}$.

We will now show that Fourier transform of the product of two functions f and g has a nice formula, that applies transforms of functions f and g. We emphasize niceness of the formula, because there is a trivial expression, that applies inverse transformation, multiplication and transformation. The formula enables us to show estimates, that would be difficult to establish otherwise. The existence of the formula is nontrivial and it does not seem to be true for Laplace transform. Separation of convolution is the most well-known property that is true for both Laplace and Fourier transforms.

Lemma 1.57. *Assume functions f and g such that we have equations*

$$\begin{aligned} f &= \mathscr{F}^{-1}\mathscr{F}f \\ g &= \mathscr{F}^{-1}\mathscr{F}g \\ \mathscr{F}f * \mathscr{F}g &= \mathscr{F}\mathscr{F}^{-1}(\mathscr{F}f * \mathscr{F}g). \end{aligned}$$

In addition, define

$$h(\omega, \omega', t) = \mathscr{F}f(\omega')\mathscr{F}g(\omega - \omega')e^{it\cdot\omega}$$

and assume

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(\omega, \omega', t)d\omega' d\omega = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(\omega, \omega', t)d\omega d\omega'.$$

Then we have the equation

$$\mathscr{F}(fg) = \mathscr{F}f * \mathscr{F}g.$$

Proof: We calculate

$$\begin{aligned} \int_{\mathbb{R}^d} \mathscr{F}f * \mathscr{F}g(t)e^{it\cdot\omega}d\omega &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathscr{F}f(\omega')\mathscr{F}g(\omega - \omega')d\omega' e^{it\cdot\omega}d\omega \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathscr{F}f(\omega')\mathscr{F}g(\omega - \omega')e^{it\cdot\omega}d\omega' d\omega \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathscr{F}f(\omega')\mathscr{F}g(\omega - \omega')e^{it\cdot\omega}d\omega d\omega' \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathscr{F}f(\omega')e^{it\cdot\omega'}\mathscr{F}g(\omega - \omega')e^{it\cdot(\omega-\omega')}d\omega d\omega' \\ &= \int_{\mathbb{R}^d} \mathscr{F}f(\omega')e^{it\cdot\omega'}\int_{\mathbb{R}^d} \mathscr{F}g(\omega - \omega')e^{it\cdot(\omega-\omega')}d\omega d\omega' \\ &= \int_{\mathbb{R}^d} \mathscr{F}f(\omega')e^{it\cdot\omega'}\int_{\mathbb{R}^d} \mathscr{F}g(\omega)e^{it\cdot\omega}d\omega d\omega' \\ &= \int_{\mathbb{R}^d} \mathscr{F}g(\omega)e^{it\cdot\omega}d\omega \int_{\mathbb{R}^d} \mathscr{F}f(\omega')e^{it\cdot\omega'}d\omega' \\ &= \int_{\mathbb{R}^d} \mathscr{F}f(\omega')e^{it\cdot\omega'}d\omega' \int_{\mathbb{R}^d} \mathscr{F}g(\omega)e^{it\cdot\omega}d\omega. \qquad (1.18) \end{aligned}$$

Equation (1.18) shows that Fourier transform separates convolution into product form. We calculate further

$$\begin{aligned} \mathscr{F}^{-1}(\mathscr{F}f * \mathscr{F}g)(t) &= \int_{\mathbb{R}^d} \mathscr{F}f * \mathscr{F}g(t)e^{it\cdot\omega}d\omega \\ &= \int_{\mathbb{R}^d} \mathscr{F}f(\omega)e^{it\cdot\omega}d\omega \int_{\mathbb{R}^d} \mathscr{F}g(\omega')e^{it\cdot\omega'}d\omega' \\ &= \mathscr{F}^{-1}\mathscr{F}f(t)\mathscr{F}^{-1}\mathscr{F}g(t) = f(t)g(t) \\ &= fg(t) \end{aligned}$$

$$\begin{aligned} fg &= \mathscr{F}^{-1}(\mathscr{F}f * \mathscr{F}g) \\ \mathscr{F}(fg) &= \mathscr{F}\mathscr{F}^{-1}(\mathscr{F}f * \mathscr{F}g) = \mathscr{F}f * \mathscr{F}g \qquad (1.19) \end{aligned}$$

and obtain the claim. □

Transformation of product is introduced in [1] with a slightly different definition of Fourier transform. The equation (1.19) now shows that transform of product is convolution of transforms.

Chapter 2

Formulation of Navier-Stokes equations and projection of the system to transversal part

2.1 The equation system

The equation system defining the flow of viscous fluid is

$$\begin{aligned} \nabla \cdot v &= 0 && (2.1)\\ \rho[v_t + (v \cdot \nabla)v] &= \mu \Delta v - \nabla p + F && (2.2) \end{aligned}$$

where $v = (v_1, v_2, v_3)$ and p are unknown functions of variables $(t, x) \in \mathbb{R} \times \mathbb{R}^3$. The system is introduced in [1]. The equation (2.1) is incompressibility condition and the equation (2.2) the Navier-Stokes equation. Equations (2.1) and (2.2) are Navier-Stokes equations. We define $\dot{v} = v_t$, $F = 0$, $\nu = \frac{\mu}{\rho}$ and divide both sides of (2.2) by ρ to obtain

$$\begin{aligned} \nabla \cdot v &= 0\\ \dot{v} + (v \cdot \nabla)v &= \nu \Delta v - \frac{1}{\rho} \nabla p. \end{aligned}$$

We are familiar with initial value problems of first order linear differential equations and we prefer writing Navier-Stokes equations in the form of an initial value problem in order to find similarities with the theory of linear equations. We obtain

$$\begin{cases} \dot{v} + (v \cdot \nabla)v = \nu \Delta v - \frac{1}{\rho} \nabla p, \quad t > 0\\ v(x, 0) = v_0(x)\\ \nabla \cdot v = 0. \end{cases} \qquad (2.3)$$

Here $v : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ is a velocity vector field and $p : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$ pressure field. The equations (2.3) are Navier-Stokes equations. We assume that for every $v_0 \in \mathscr{S}_3^3$ the system (2.3) has a smooth solution (v, p) such that for fixed $t \in \mathbb{R}$ the function $v(x, t)$ is an element of $\mathscr{S}_3^3$ and the function $p(x, t)$ an element of $\mathscr{S}_3$. In addition, we assume that fields v and $\mathscr{F}_x v$ decrease to zero pointwise as time goes to infinity.

2.2 Derivation of projections

From basic vector calculus we know that any sufficiently smooth field has a decomposition to irrotational and sourceless parts. Chapter 12 of [4] shows this property with certain assumptions. Helmholtz's theorem applies the decomposition.

We recall that a field v is sourceless if it satisfies

$$\nabla \cdot v = 0 \tag{2.4}$$

and irrotational, if it satisfies

$$\nabla \times v = 0. \tag{2.5}$$

We will derive expressions, that allow us to calculate the decomposition. Take any v satisfying sufficient assumptions. Then $\mathscr{F}v$ has a unique decomposition to real and imaginary parts. Both of them have a pointwise unique decomposition parallel and perpendicular parts with respect to wave vector k. We calculate

$$\begin{aligned}
\mathscr{F}(\nabla \cdot v)(k) &= \mathscr{F}\sum_{n=1}^{3} \partial_n[v_n](k) = \sum_{n=1}^{3} \mathscr{F}\partial_n[v_n](k) = \sum_{n=1}^{3} ik_n \mathscr{F}[v_n](k) \\
&= \sum_{n=1}^{3} ik_n \mathscr{F}v_n(k) = \sum_{n=1}^{3} ik_n \mathscr{F}v(k)_n = ik \cdot \mathscr{F}v(k)
\end{aligned}$$

$$\begin{aligned}
\mathscr{F}(\nabla \times v)(k) &= \mathscr{F}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \end{vmatrix}(k) \\
&= \mathscr{F}((\partial_2 v_3 - \partial_3 v_2) - (\partial_1 v_3 - \partial_3 v_1) + (\partial_1 v_2 - \partial_2 v_1))(k) \\
&= \mathscr{F}(\partial_2 v_3 - \partial_3 v_2)(k) - \mathscr{F}(\partial_1 v_3 - \partial_3 v_1)(k) \\
&\quad + \mathscr{F}(\partial_1 v_2 - \partial_2 v_1)(k) \\
&= (\mathscr{F}\partial_2 v_3(k) - \mathscr{F}\partial_3 v_2(k)) - (\mathscr{F}\partial_1 v_3(k) - \mathscr{F}\partial_3 v_1(k)) \\
&\quad + (\mathscr{F}\partial_1 v_2(k) - \mathscr{F}\partial_2 v_1(k)) \\
&= (ik_2 \mathscr{F}v_3(k) - ik_3 \mathscr{F}v_2(k)) - (ik_1 \mathscr{F}v_3(k) - ik_3 \mathscr{F}v_1(k)) \\
&\quad + (ik_1 \mathscr{F}v_2(k) - ik_2 \mathscr{F}v_1(k)) \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ ik_1 & ik_2 & ik_3 \\ \mathscr{F}v_1(k) & \mathscr{F}v_2(k) & \mathscr{F}v_3(k) \end{vmatrix} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ ik_1 & ik_2 & ik_3 \\ (\mathscr{F}v)_1(k) & (\mathscr{F}v)_2(k) & (\mathscr{F}v)_3(k) \end{vmatrix} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (ik)_1 & (ik)_2 & (ik)_3 \\ \mathscr{F}v(k)_1 & \mathscr{F}v(k)_2 & \mathscr{F}v(k)_3 \end{vmatrix} = ik \times \mathscr{F}v(k)
\end{aligned}$$

The parallel part satisfies

$$\begin{aligned}
ik \times \mathscr{F}v(k) &= ik \times (\mathrm{Re}(\mathscr{F}v(k)) + i\mathrm{Im}(\mathscr{F}v(k))) \\
&= ik \times \mathrm{Re}(\mathscr{F}v(k)) + ik \times i\mathrm{Im}(\mathscr{F}v(k)) \\
&= ik \times \mathrm{Re}(\mathscr{F}v(k)) - k \times \mathrm{Im}(\mathscr{F}v(k)) = 0.
\end{aligned}$$

The perpendicular part satisfies

$$\begin{aligned}
ik \cdot \mathscr{F}v(k) &= ik \cdot (\mathrm{Re}(\mathscr{F}v(k)) + i\mathrm{Im}(\mathscr{F}v(k))) \\
&= ik \cdot \mathrm{Re}(\mathscr{F}v(k)) + ik \cdot i\mathrm{Im}(\mathscr{F}v(k)) \\
&= ik \cdot \mathrm{Re}(\mathscr{F}v(k)) - k \cdot \mathrm{Im}(\mathscr{F}v(k)) \\
&= 0 \qquad (2.6)
\end{aligned}$$

We map both sides of both equations with $\mathscr{F}^{-1}$ to obtain equations (2.4) and (2.5).

We define maps P_1 and P_2 from $\mathscr{S}(\mathbb{R}^3)$ to $C(\mathbb{R}^3)$ so that $P_1 v$ is the longitudinal part of v and $P_2 v$ the transversal part of v.

We derive expressions to these operators. We apply the projections to parallel and perpendicular parts separately to real and imaginary parts of $\mathscr{F}v$. Linearity of the projection expressions allows us to calculate projections directly from the complex form. That is why we assume that $b \in \mathbb{R}^3$ during derivation, but allow compex vectors after the derivation is completed. Fix $x = \mathscr{F}v \in V^3$ and $k \in S$. Consider now vectors $a = k$ and $b = x(k)$. We have

$$a \cdot b = |a||b|\cos(\alpha).$$

For $b = 0$ the parallel projection of b to a is 0. For $b \neq 0$ we calculate

$$b_1 = |b|\cos(\alpha)\frac{a}{|a|} = |b|\frac{a \cdot b}{|a||b|}\frac{a}{|a|} = \frac{a \cdot b}{|a|^2}a = a\frac{1}{|a|^2}a \cdot b.$$

For $b = 0$ the perpendicular projection of b with respect to a is 0. For $b \neq 0$ we calculate

$$b_2 = b - b_1 = b - a\frac{1}{|a|^2}a \cdot b.$$

For complex vectors $b \in \mathbb{C}^3$ we calculate

$$\begin{aligned}
b_1 &= a\frac{1}{|a|^2}a \cdot \mathrm{Re}(b) + ia\frac{1}{|a|^2}a \cdot \mathrm{Im}(b) = a\frac{1}{|a|^2}a \cdot (\mathrm{Re}(b) + i\mathrm{Im}(b)) \\
&= a\frac{1}{|a|^2}a \cdot b \\
b_2 &= \mathrm{Re}(b) - \mathrm{Re}(b)_1 + i(\mathrm{Im}(b) - \mathrm{Im}(b)_1) \\
&= \mathrm{Re}(b) + i\mathrm{Im}(b) - (\mathrm{Re}(b)_1 + i\mathrm{Im}(b)_1) \\
&= b - b_1 = b - a\frac{1}{|a|^2}a \cdot b.
\end{aligned}$$

We recall that a was assumed to be fixed. We define mappings

$$T_1' : S \times \mathbb{C}^3 \to \mathbb{C}^3, \quad T_1'(a,b) = a\frac{1}{|a|^2}a \cdot b \qquad (2.7)$$

$$T_2' : S \times \mathbb{C}^3 \to \mathbb{C}^3, \quad T_2'(a,b) = b - a\frac{1}{|a|^2}a \cdot b \qquad (2.8)$$

to obtain

$$\begin{aligned}
T_1'(k, x(k)) &= k\frac{1}{|k|^2}k \cdot x(k) \\
T_2'(k, x(k)) &= x(k) - k\frac{1}{|k|^2}k \cdot x(k)
\end{aligned}$$

for fixed $x \in V^3$ and $k \in S$. We define operators

$$\begin{aligned} T_1 : V^3 \to V^3, \quad (T_1 x)(k) &= T_1'(k, x(k)) \\ T_2 : V^3 \to V^3, \quad (T_2 x)(k) &= T_2'(k, x(k)). \end{aligned}$$

and obtain

$$T_1 x(k) = k \frac{1}{|k|^2} k \cdot x(k) \tag{2.9}$$

$$T_2 x(k) = x(k) - k \frac{1}{|k|^2} k \cdot x(k)\,. \tag{2.10}$$

Operators T_1 and T_2 clearly satisfy

$$T_1 + T_2 = I,$$

where $I : V^3 \to V^3$ is identity operator. Under sufficient smoothness and integrability conditions we now define

$$P_1' = \mathscr{F}^{-1} T_1 \mathscr{F} \tag{2.11}$$

$$P_2' = \mathscr{F}^{-1} T_2 \mathscr{F}. \tag{2.12}$$

Corollary 1.14 shows that the Laplacian Δ has right inverse. Assume that v is a scalar field. We calculate

$$\begin{aligned} \mathscr{F}\Delta v(k) &= \left[\mathscr{F}\sum_{n=1}^{3}\partial_n^2 v\right](k) = \left[\sum_{n=1}^{3}\mathscr{F}\partial_n^2 v\right](k) = \sum_{n=1}^{3}\mathscr{F}\partial_n^2 v(k) \\ &= \sum_{n=1}^{3}\mathscr{F}\partial_n\partial_n v(k) = \sum_{n=1}^{3} ik_n \mathscr{F}\partial_n v(k) \\ &= \sum_{n=1}^{3} ik_n ik_n \mathscr{F} v(k) = \sum_{n=1}^{3}\mathscr{F} v(k) ik_n ik_n \\ &= \sum_{n=1}^{3}\mathscr{F} v(k)(ik_n)^2 = \sum_{n=1}^{3}\mathscr{F} v(k) i^2 k_n^2 \\ &= \mathscr{F} v(k) i^2 \sum_{n=1}^{3} k_n^2 = \mathscr{F} v(k) i^2 |k|^2 = i^2 |k|^2 \mathscr{F} v(k) \\ &= -|k|^2 \mathscr{F} v(k) \end{aligned}$$

$$\mathscr{F}\nabla v = \mathscr{F}\sum_{m=1}^{3}\partial_m v e_m = \sum_{m=1}^{3}\mathscr{F}(\partial_m v e_m) = \sum_{m=1}^{3}\mathscr{F}\partial_m v e_m$$

$$\begin{aligned} \mathscr{F}\nabla v(k) &= \left[\sum_{m=1}^{3}\mathscr{F}\partial_m v e_m\right](k) = \sum_{m=1}^{3}\left[\mathscr{F}\partial_m v e_m\right](k) \\ &= \sum_{m=1}^{3}\mathscr{F}\partial_m v(k) e_m = \sum_{m=1}^{3} ik_m \mathscr{F} v(k) e_m \\ &= \sum_{m=1}^{3}\mathscr{F} v(k) ik_m e_m = \mathscr{F} v(k) i \sum_{m=1}^{3} k_m e_m \end{aligned}$$

$$= \mathscr{F}v(k)ik = ik\mathscr{F}v(k)$$

We now obtain

$$\begin{aligned}
\mathscr{F}(\nabla\Delta^{-1}\nabla\cdot v)(k) &= ik\mathscr{F}(\Delta^{-1}\nabla\cdot v)(k) = ik\frac{-1}{|k|^2}(-|k|^2)\mathscr{F}(\Delta^{-1}\nabla\cdot v)(k) \\
&= ik\frac{-1}{|k|^2}\mathscr{F}(\Delta\Delta^{-1}\nabla\cdot v)(k) = ik\frac{-1}{|k|^2}\mathscr{F}(\nabla\cdot v)(k) \\
&= ik\frac{-1}{|k|^2}ik\cdot\mathscr{F}v(k) = k\frac{1}{|k|^2}k\cdot\mathscr{F}v(k)
\end{aligned}$$

$$\begin{aligned}
\mathscr{F}P_1'v(k) &= \mathscr{F}\mathscr{F}^{-1}T_1\mathscr{F}v(k) = T_1\mathscr{F}v(k) = k\frac{1}{|k|^2}k\cdot\mathscr{F}v(k) \\
&= \mathscr{F}(\nabla\Delta^{-1}\nabla\cdot v)(k) \\
P_1'v &= \mathscr{F}^{-1}\mathscr{F}P_1'v = \mathscr{F}^{-1}\mathscr{F}\nabla\Delta^{-1}\nabla\cdot v = \nabla\Delta^{-1}\nabla\cdot v \qquad (2.13)
\end{aligned}$$

$$\begin{aligned}
\mathscr{F}P_2'v(k) &= \mathscr{F}\mathscr{F}^{-1}T_2\mathscr{F}v(k) = T_2\mathscr{F}v(k) = \mathscr{F}v(k) - k\frac{1}{|k|^2}k\cdot\mathscr{F}v(k) \\
&= \mathscr{F}v(k) - \mathscr{F}(\nabla\Delta^{-1}\nabla\cdot v)(k) = (\mathscr{F}v - \mathscr{F}\nabla\Delta^{-1}\nabla\cdot v)(k) \\
\mathscr{F}P_2'v &= \mathscr{F}v - \mathscr{F}\nabla\Delta^{-1}\nabla\cdot v = \mathscr{F}(v - \nabla\Delta^{-1}\nabla\cdot v)
\end{aligned}$$

$$\begin{aligned}
P_2'v &= \mathscr{F}^{-1}\mathscr{F}P_2'v = \mathscr{F}^{-1}\mathscr{F}(v - \nabla\Delta^{-1}\nabla\cdot v) \\
&= v - \nabla\Delta^{-1}\nabla\cdot v \qquad (2.14)
\end{aligned}$$

We now show that operators P_1 and P_2 decompose any sufficiently smooth and integrable v to irrotational and sourceless parts. We have

$$P_1'v + P_2'v = \nabla\Delta^{-1}\nabla\cdot v + v - \nabla\Delta^{-1}\nabla\cdot v = v \qquad (2.15)$$

and find that the left hand side is a decomposition of v. We recall that the curl of any gradient field is zero and obtain

$$\begin{aligned}
\nabla\cdot P_2'v &= \nabla\cdot(v - \nabla\Delta^{-1}\nabla\cdot v) = \nabla\cdot v - \nabla\cdot\nabla\Delta^{-1}\nabla\cdot v \\
&= \nabla\cdot v - \Delta\Delta^{-1}\nabla\cdot v = \nabla\cdot v - \nabla\cdot v = 0 \qquad (2.16) \\
\nabla\times P_1'v &= \nabla\times\nabla\Delta^{-1}\nabla\cdot v = 0. \qquad (2.17)
\end{aligned}$$

Hence we have

$$\begin{aligned}
P_1 &= P_1' \qquad (2.18) \\
P_2 &= P_2'. \qquad (2.19)
\end{aligned}$$

Hereafter we call them longitudinal and transversal parts ov v. The operator P_2 is physically important because sourceless heat field, flow field and electromagnetic field satisfy

$$v = v - \nabla\Delta^{-1}\nabla\cdot v = P_2'v = P_2v. \qquad (2.20)$$

Operators P_1 and P_2 have basic properties

$$\begin{aligned}
P_1v &= P_1'v = \nabla\Delta^{-1}\nabla\cdot v \\
P_2v &= P_2'v = v - \nabla\Delta^{-1}\nabla\cdot v
\end{aligned}$$

$$
\begin{aligned}
(P_1+P_2)v &= P_1v+P_2v = P_1'v+P_2'v = v = Iv\\
P_1+P_2 &= I.
\end{aligned}
$$

We calculate further

$$
\begin{aligned}
P_1P_1v &= \nabla\Delta^{-1}\nabla\cdot\nabla\Delta^{-1}\nabla\cdot v = \nabla\Delta^{-1}\Delta\Delta^{-1}\nabla\cdot v = \nabla\Delta^{-1}\nabla\cdot v = P_1v\\
P_1P_1 &= P_1
\end{aligned}
$$

$$
\begin{aligned}
P_2P_2 &= (I-P_1)(I-P_1) = I(I-P_1) - P_1(I-P_1)\\
&= II - IP_1 - P_1I + P_1P_1\\
&= I - P_1 - P_1 + P_1 = I - P_1 = P_2
\end{aligned}
$$

$$
\begin{aligned}
P_1^2 &= P_1P_1 = P_1\\
P_2^2 &= P_2P_2 = P_2.
\end{aligned}
$$

This shows that operators P_1 and P_2 are projections. For linear operators L we define

$$\hat{L} = \mathscr{F} L \mathscr{F}^{-1}$$

whenever the right hand side exists. Operators P_1 and P_2 have representations

$$
\begin{aligned}
\hat{P}_1 &= \mathscr{F}P_1\mathscr{F}^{-1} = \mathscr{F}P_1'\mathscr{F}^{-1} = \mathscr{F}\mathscr{F}^{-1}T_1\mathscr{F}\mathscr{F}^{-1} = T_1 && (2.21)\\
\hat{P}_2 &= \mathscr{F}P_2\mathscr{F}^{-1} = \mathscr{F}P_2'\mathscr{F}^{-1} = \mathscr{F}\mathscr{F}^{-1}T_2\mathscr{F}\mathscr{F}^{-1} = T_2. && (2.22)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\hat{P}_1x(k) &= T_1x = k\frac{1}{|k|^2}k\cdot x(k), \quad k\neq 0\\
\hat{P}_2x(k) &= T_2x = x(k) - k\frac{1}{|k|^2}k\cdot x(k), \quad k\neq 0.
\end{aligned}
$$

For timespace fields we write projections Π^I and Π^S such that for fixed time we apply projections P_1 and P_2. For timespace field operators L we define

$$\hat{L} = \mathscr{F}_x L \mathscr{F}_x^{-1}.$$

We define

$$
\begin{aligned}
\hat{\Pi}^I u(k,t) &= k\frac{1}{|k|^2}k\cdot u(k,t), \quad k\neq 0,\\
\hat{\Pi}^S u(k,t) &= u(k,t) - k\frac{1}{|k|^2}k\cdot u(k,t), \quad k\neq 0.
\end{aligned}
$$

We define

$$
\begin{aligned}
\hat{\Pi}^I(k)x &= k\frac{1}{|k|^2}k\cdot x, \quad k\neq 0,\\
\hat{\Pi}^S(k)x &= x - k\frac{1}{|k|^2}k\cdot x, \quad k\neq 0,
\end{aligned}
$$

where $x\in\mathbb{C}^3$. Hence we have

$$
\begin{aligned}
\hat{\Pi}^I u(k,t) &= \hat{\Pi}^I(k)u(k,t), \quad k\neq 0,\\
\hat{\Pi}^S u(k,t) &= \hat{\Pi}^S(k)u(k,t), \quad k\neq 0.
\end{aligned}
$$

Operators $\hat{\Pi}^I(k)$ and $\hat{\Pi}^S(k)$ clearly share the same projection properties with T_1 and T_2. They are also clearly linear.

2.3 Boundedness estimates

We calculate

$$\begin{aligned}
|\hat{\Pi}^I(k)x| &= \left|k\frac{1}{|k|^2}k\cdot x\right| \le |k|\frac{1}{|k|^2}||k\cdot x| \le |k|\frac{1}{|k|^2}|k||x| = |x| \\
|\hat{\Pi}^S(k)x| &= \left|x - k\frac{1}{|k|^2}k\cdot x\right| \le \left|x\right| + \left|k\frac{1}{|k|^2}k\cdot x\right| \\
&= |x| + |k|\frac{1}{|k|^2}|k||x| = 2|x| \\
|\hat{\Pi}^I(k)| &= \sup_{|x|=1}|\hat{\Pi}^I(k)x| \le \sup_{|x|=1}|x| = 1 \\
|\hat{\Pi}^S(k)| &= \sup_{|x|=1}|\hat{\Pi}^S(k)x| \le \sup_{|x|=1}2|x| = 2.
\end{aligned}$$

This shows, that operators $\hat{\Pi}^I(k)$ and $\hat{\Pi}^S(k)$ are bounded for every $k \in S$.

2.4 Projection of Navier-Stokes equations to transversal part

In this section we project the system (2.3) into equation system with one equation and initial value. The projected system will be of interest in the examination of mathematical properties of Navier-Stokes equations. In Section 2 we introduced the incompressible Navier-Stokes equations (2.3). We now use projections P_1 and P_2 and find that pressure is a consequence of flow and acts as a constraint force arising from the incompressibility of the flow.

Navier-Stokes equations are

$$\left\{\begin{array}{lcll}
\dot{v} + (v\cdot\nabla)v &=& \nu\Delta v - \frac{1}{\rho}\nabla p, & t>0 \\
v(x,0) &=& v_0(x) & \\
\nabla\cdot v &=& 0, & t>0.
\end{array}\right. \tag{2.23}$$

We apply projections Π^I and Π^S to both sides of (2.3) and obtain

$$\left\{\begin{array}{lcl}
\Pi^I\partial_t v + \Pi^I(v\cdot\nabla)v &=& \Pi^I\nu\Delta v - \frac{1}{\rho}\Pi^I\nabla p \\
\Pi^S\partial_t v + \Pi^S(v\cdot\nabla)v &=& \Pi^S\nu\Delta v - \frac{1}{\rho}\Pi^S\nabla p \\
\Pi^I v(x,0) &=& \Pi^I v_0 \\
\Pi^S v(x,0) &=& \Pi^S v_0 \\
\nabla\cdot v &=& 0.
\end{array}\right. \tag{2.24}$$

Under sufficient smoothness and integrability assumptions we have

$$\begin{aligned}
\partial_t\Pi^S &= \Pi^S\partial_t \\
\Delta\Pi^I &= \Pi^I\Delta \\
\Delta\Pi^S &= \Pi^S\Delta.
\end{aligned}$$

Hence

$$\left\{\begin{array}{lcl}
\partial_t\Pi^I v + \Pi^I(v\cdot\nabla)v &=& \nu\Delta\Pi^I v - \frac{1}{\rho}\Pi^I\nabla p \\
\partial_t\Pi^S v + P_2(v\cdot\nabla)v &=& \nu\Delta\Pi^S v - \frac{1}{\rho}\Pi^S\nabla p \\
\Pi^I v(x,0) &=& \Pi^I v_0 \\
\Pi^S v(x,0) &=& \Pi^S v_0 \\
\nabla\cdot v &=& 0.
\end{array}\right. \tag{2.25}$$

Now incompressibility implies $\Pi^I v = 0$ and $\Pi^S v = v$. The field ∇p is gradient field and hence longitudinal. We obtain

$$\begin{aligned}\Pi^S \nabla p &= \nabla p - \nabla\Delta^{-1}\nabla\cdot\nabla p = \nabla p - \nabla p = 0,\\ \Pi^I \nabla p &= \nabla\Delta^{-1}\nabla\cdot\nabla p = \nabla\Delta^{-1}\Delta p = \nabla p.\end{aligned}$$

Hence

$$\begin{cases}\frac{1}{\rho}\nabla p &= -\nabla\Delta^{-1}\nabla\cdot(v\cdot\nabla)v\\ \partial_t v - \nu\Delta v &= -\Pi^S(v\cdot\nabla)v\\ 0 &= 0\\ v(x,0) &= v_0.\end{cases} \tag{2.26}$$

The choise $p = -\rho\Delta^{-1}\nabla\cdot(v\cdot\nabla)v$ satisfies the first equation and we obtain

$$\begin{cases}p &= -\rho\Delta^{-1}\nabla\cdot(v\cdot\nabla)v\\ \partial_t v - \nu\Delta v &= -\Pi^S(v\cdot\nabla)v\\ v(x,0) &= v_0(x).\end{cases} \tag{2.27}$$

We solve the left hand side heat equation of the second equation of (2.27) using remark (17) in Section 2.3.1 in [2] to obtain

$$\begin{aligned}v(x,t) &= \int_{\mathbb{R}^n} v(x',0)\frac{1}{\sqrt{4\pi\nu t}^n}e^{-\frac{|x-x'|^2}{4\nu t}}dx'\\ &\quad+\int_{\mathbb{R}^n}\int_0^t -P_2(v\cdot\nabla)v(x',\tau)\frac{1}{\sqrt{4\pi\nu(t-\tau)}^n}e^{-\frac{|x-x'|^2}{4\nu(t-\tau)}}d\tau dx'\\ &= \int_{\mathbb{R}^n} v_0(x')\frac{1}{\sqrt{4\pi\nu t}^n}e^{-\frac{|x-x'|^2}{4\nu t}}dx'\\ &\quad+\int_{\mathbb{R}^n}\int_0^t -P_2(v\cdot\nabla)v(x',\tau)\frac{1}{\sqrt{4\pi\nu(t-\tau)}^n}e^{-\frac{|x-x'|^2}{4\nu(t-\tau)}}d\tau dx'\\ &= \int_{\mathbb{R}^n} v_0(x-x')\frac{1}{\sqrt{4\pi\nu t}^n}e^{-\frac{|x'|^2}{4\nu t}}dx'\\ &\quad+\int_{\mathbb{R}^n}\int_0^t -P_2(v\cdot\nabla)v(x-x',t-\tau)\frac{1}{\sqrt{4\pi\nu\tau}^n}e^{-\frac{|x'|^2}{4\nu\tau}}d\tau dx'\end{aligned} \tag{2.28}$$

for $t > 0$, where $n = 3$. This shows that v is sourcesless as an integral of sourceless fields. We take gradients sidewise from the first equation of (2.27), divide by ρ and add the result to the second equation to obtain the first equation of (2.3). The last equation of (2.3) holds because v is sourceless and hence we see that solutions of (2.27) satisfy also the original system (2.3).

We have shown that systems (2.3) and (2.27) can be solved using the reduced system

$$\begin{cases}\dot{v} - \nu\Delta v &= -\Pi^S(v\cdot\nabla)v\\ v(x,0) &= v_0(x).\end{cases} \tag{2.29}$$

Hence the original system (2.3) has an equivalent formulation (2.29).

Chapter 3

Transformation of the reduced equation system

3.1 Some necessary definitions

In this section we transform the equation system (2.29) and obtain a single equation $u = Tu + g$. We also prove some estimates for T.

Definition 3.1. Assume function matrix $A \in W^{n \times n}$ and function vector $x \in W^n$. We define multiplication $\cdot : W^{n \times n} \times W^n \to W^n$ by

$$[A \cdot x]_i = \sum_j [A]_{i,j} [x]_j.$$

For notational convenience we write Ax instead of $A \cdot x$.

Definition 3.2. Function matrix $A \in W^{n \times n}$ is evaluated elementwise, that is

$$[A(k, \omega)]_{i,j} = [A]_{i,j}(k, \omega).$$

Function vector $x \in W^n$ is evaluated elementwise, that is

$$[x(k, \omega)]_i = [x]_i(k, \omega).$$

Definition 3.3. N-dimensional convolution $f * g$ of two vector functions $f \in W^n$ and $g \in W^n$ is defined by

$$[f * g]_{ij} = f_i * g_j$$

whenever all convolutions exist.

Definition 3.4. Supremum-norm of a function $f \in W^n$ is defined by

$$||f|| = \sup_{k \in S, \omega \in \mathbb{R}} |f(k, \omega)|.$$

Definition 3.5. L^1-norm of a function $f \in W^n$ is defined by

$$||f||_1 = \int_S \int_{\mathbb{R}} |f(k, \omega)| dk \, d\omega.$$

3.2 Transformation over space

In order to be able to write a Fourier analysis for Navier-Stokes equations we transform them. Consider the equation system

$$\begin{cases} \dot{v} - \nu\Delta v &= -\Pi^S(v\cdot\nabla)v \\ v(x,0) &= v_0(x), \quad v_0 \in \mathscr{S}_3, \quad \nabla\cdot v_0 = 0. \end{cases}$$

We calculate to obtain

$$\begin{aligned} (v\cdot\nabla)v &= \sum_{n=1}^{3}(v\cdot\nabla)v_n e_n = \sum_{n=1}^{3}(\sum_{m=1}^{3} v_m\partial_m)v_n e_n = \sum_{n=1}^{3}(\sum_{m=1}^{3} v_m\partial_m v_n)e_n \\ &= \sum_{n=1}^{3}\sum_{m=1}^{3} v_m\partial_m v_n e_n. \\ \nabla\cdot v &= \sum_{n=1}^{3}\partial_n v_n \\ \Delta v &= \sum_{n=1}^{3}\partial_n^2 v = \sum_{n=1}^{3}\partial_n\partial_n v. \end{aligned}$$

Hence we have

$$\begin{cases} \dot{v} - \nu\sum_{n=1}^{3}\partial_n\partial_n v &= -\Pi^S\sum_{n=1}^{3}\sum_{m=1}^{3} v_m\partial_m v_n e_n \\ \sum_{n=1}^{3}\partial_n v_n &= 0 \\ v(x,0) &= v_0(x), \quad v_0\in\mathscr{S}_3, \quad \sum_{n=1}^{3}\partial_n v_{0n} = 0 \end{cases}$$

where e_k are basis vectors of $\mathbb{R}^3$.

We calculate

$$\begin{aligned} \sum_{m=1}^{3} v_m\partial_m v_n &= 0v_n + \sum_{m=1}^{3} v_m\partial_m v_n = \left[\sum_{m=1}^{3}\partial_m v_m\right]v_n + \sum_{m=1}^{3} v_m\partial_m v_n \\ &= \sum_{m=1}^{3}\partial_m v_m v_n + \sum_{m=1}^{3} v_m\partial_m v_n \\ &= \sum_{m=1}^{3}(\partial_m v_m v_n + v_m\partial_m v_n) = \sum_{m=1}^{3}\partial_m(v_m v_n) \end{aligned}$$

and obtain

$$\begin{cases} \dot{v} - \nu\sum_{n=1}^{3}\partial_n\partial_n v &= -\Pi^S\sum_{n=1}^{3}\sum_{m=1}^{3}\partial_m(v_m v_n)e_n \quad |\mathscr{F}_x \\ \sum_{n=1}^{3}\partial_n v_n &= 0 \\ v(x,0) &= v_0(x), \quad v_0\in\mathscr{S}, \quad \sum_{k=1}^{3}\partial_k v_{0k} = 0. \end{cases}$$

We transform both sides of the system over space with Fourier transform

$$\mathscr{F}_x v(k,t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} v(x,t)e^{-ik\cdot x}dx$$

where $k \in S$. Set $u = \mathscr{F}_x v$, $u_0 = \mathscr{F}_x v_0$ and $q = \mathscr{F}_x p$. We calculate

$$\begin{aligned}
\mathscr{F}_x v_j &= (\mathscr{F}_x v)_j = u_j \\
\mathscr{F}_x v_{0j} &= (\mathscr{F}_x v_0)_j = u_{0j}.
\end{aligned}$$

For any scalar field u satisfying sufficient assumptions we calculate

$$\begin{aligned}
(ue_n)(k,t) &= \sum_{m=1}^{3} (ue_n)(k,t)_m e_m = \sum_{m=1}^{3} (ue_n)_m(k,t) e_m \\
&= \sum_{m=1}^{3} (u(e_n)_m)(k,t) e_m = \sum_{m=1}^{3} (u\delta_{n,m})(k,t) e_m \\
&= \sum_{m=1}^{3} (\delta_{n,m} u)(k,t) e_m = \sum_{m=1}^{3} \delta_{n,m} u(k,t) e_m \\
&= u(k,t) e_n
\end{aligned}$$

$$\left[\sum_{n=1}^{3} u_n e_n\right](k,t) = \sum_{n=1}^{3} (u_n e_n)(k,t) = \sum_{n=1}^{3} u_n(k,t) e_n$$

We transform the convection term to obtain

$$\begin{aligned}
\mathscr{F}_x \Pi^S \sum_{n=1}^{3}\sum_{m=1}^{3} \partial_m (v_m v_n) e_n &= \mathscr{F}_x \Pi^S \mathscr{F}_x^{-1} \mathscr{F}_x \sum_{n=1}^{3}\sum_{m=1}^{3} \partial_m (v_m v_n) e_n \\
&= \hat{\Pi}^S \mathscr{F}_x \sum_{n=1}^{3}\sum_{m=1}^{3} \partial_m (v_m v_n) e_n \\
&= \hat{\Pi}^S \sum_{n=1}^{3} \left[\mathscr{F}_x \sum_{m=1}^{3} \partial_{m=1}^{3} (v_m v_n)\right] e_n \\
&= \hat{\Pi}^S \sum_{n=1}^{3}\sum_{m=1}^{3} \mathscr{F}_x \partial_m (v_m v_n) e_n.
\end{aligned}$$

We calculate further and obtain

$$\begin{aligned}
\mathscr{F}_x \left[\Pi^S \sum_{n=1}^{3}\sum_{m=1}^{3} \partial_m (v_m v_n) e_n\right](k,t) &= \left[\hat{\Pi}^S \sum_{n=1}^{3}\sum_{m=1}^{3} \mathscr{F}_x \partial_m (v_m v_n) e_n\right](k,t) \\
&= \hat{\Pi}^S(k) \sum_{n=1}^{3}\sum_{m=1}^{3} \mathscr{F}_x \partial_m (v_m v_n) e_n\Big](k,t) \\
&= \hat{\Pi}^S(k) \sum_{n=1}^{3} \left[\sum_{m=1}^{3} \mathscr{F}_x \partial_m (v_m v_n)\right](k,t) e_n \\
&= \hat{\Pi}^S(k) \sum_{n=1}^{3}\sum_{m=1}^{3} \mathscr{F}_x \partial_m (v_m v_n)(k,t) e_n \\
&= \hat{\Pi}^S(k) \sum_{n=1}^{3}\sum_{m=1}^{3} i k_m \mathscr{F}_x (v_m v_n)(k,t) e_n
\end{aligned}$$

$$\begin{aligned}
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3} ik_m(\mathscr{F}_x v_m * \mathscr{F}_x v_n)(k,t)e_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3} ik_m \int_{\mathbb{R}^3} \mathscr{F}_x v_m(k',t)\mathscr{F}_x v_n(k-k',t)dk' e_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3} \mathscr{F}_x v_m(k',t)\mathscr{F}_x v_n(k-k',t)dk' ik_m e_n. \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3} u_m(k',t)u_n(k-k',t)dk' ik_m e_n.
\end{aligned}$$

We transform the Laplacian term to obtain

$$\begin{aligned}
\mathscr{F}_x \sum_{n=1}^{3} \partial_n\partial_n v(k,t) &= \sum_{n=1}^{3} \mathscr{F}_x \partial_n\partial_n v(k,t) = \sum_{n=1}^{3} ik_n \mathscr{F}_x \partial_n v(k,t) \\
&= \sum_{n=1}^{3} ik_n ik_n \mathscr{F} v(k,t) = \sum_{n=1}^{3} \mathscr{F}_x v(k,t) ik_n ik_n \\
&= \sum_{n=1}^{3} \mathscr{F}_x v(k,t)(ik_n)^2 = \sum_{n=1}^{3} \mathscr{F}_x v(k,t) i^2 k_n^2 \\
&= \mathscr{F}_x v(k,t) i^2 \sum_{n=1}^{3} k_n^2 = \mathscr{F}_x v(k,t) i^2 |k|^2 \\
&= i^2 |k|^2 \mathscr{F}_x v(k,t) = -|k|^2 \mathscr{F}_x v(k,t) \\
&= -|k|^2 u(k,t).
\end{aligned}$$

Sets $\mathbb{R}^3$ and $[t_1, t_2]$ are σ-finite measure spaces with Lebesgue measure and the latter integral in the change of order of integration is absolutely convergent. We calculate using Fubini's theorem (Th 8.8 in [11]) and the fundamental theorem of analysis to obtain

$$\begin{aligned}
&\mathscr{F}_x v(k,t_2) - \mathscr{F}_x v(k,t_1) \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} v(x,t_2)e^{-ik\cdot x}dx - \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} v(x,t_1)e^{-ik\cdot x}dx \\
&= \frac{1}{(2\pi)^3}\left[\int_{\mathbb{R}^3} v(x,t_2)e^{-ik\cdot x}dx - \int_{\mathbb{R}^3} v(x,t_1)e^{-ik\cdot x}dx\right] \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} (v(x,t_2)e^{-ik\cdot x} - v(x,t_1)e^{-ik\cdot x})dx \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} (v(x,t_2) - v(x,t_1))e^{-ik\cdot x}dx \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{t_1}^{t_2} \dot{v}(x,t)dt e^{-ik\cdot x}dx \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-ik\cdot x}\int_{t_1}^{t_2} \dot{v}(x,t)dtdx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{t_1}^{t_2} e^{-ik\cdot x}\dot{v}(x,t)dtdx = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{t_1}^{t_2} \dot{v}(x,t)e^{-ik\cdot x}dtdx \\
&= \frac{1}{(2\pi)^3}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \dot{v}(x,t)e^{-ik\cdot x}dxdt.
\end{aligned}
$$

We transform the time derivative term to obtain

$$
\begin{aligned}
\mathscr{F}_x\dot{v}(k,t) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\dot{v}(x,t)e^{-ik\cdot x}dx \\
&= \frac{1}{(2\pi)^3}\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^3}\dot{v}(x,t)e^{-ik\cdot x}dxdt \\
&= \lim_{h\to 0}\frac{1}{(2\pi)^3}\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^3}\dot{v}(x,t)e^{-ik\cdot x}dxdt \\
&= \lim_{h\to 0}\frac{1}{h}\frac{1}{(2\pi)^3}\int_t^{t+h}\int_{\mathbb{R}^3}\dot{v}(x,t)e^{-ik\cdot x}dxdt
\end{aligned}
$$

$$
= \lim_{h\to 0}\frac{\mathscr{F}_x v(k,t+h) - \mathscr{F}_x v(k,t)}{h} = \lim_{h\to 0}\frac{u(k,t+h)-u(k,t)}{h} = \dot{u}(k,t).
$$

We transform incompressibility terms to obtain

$$
\begin{aligned}
\mathscr{F}_x\sum_{n=1}^{3}\partial_n v_n(k,t) &= \sum_{n=1}^{3}\mathscr{F}_x\partial_n v_n(k,t) = \sum_{n=1}^{3} ik_n\mathscr{F}_x v_n(k,t) \\
&= \sum_{n=1}^{3} ik_n(\mathscr{F}_x v)_n(k,t) = \sum_{n=1}^{3} ik_n\mathscr{F}_x v(k,t)_n \\
&= i\sum_{n=1}^{3} k_n\mathscr{F}_x v(k,t)_n = ik\cdot\mathscr{F}_x v(k,t).
\end{aligned}
$$

We calculate further and obtain

$$
\begin{aligned}
\mathscr{F}_x\sum_{n=1}^{3}\partial_n v_n(k,t) &= ik\cdot\mathscr{F}_x v(k,t) = ik\cdot u(k,t), \\
\mathscr{F}_x\sum_{n=1}^{3}\partial_n v_{0n}(k,t) &= ik\cdot\mathscr{F}_x v_0(k,t) = ik\cdot u_0(k,t).
\end{aligned}
$$

We obtain the system

$$
\begin{aligned}
\dot{u}(k,t) + \nu|k|^2 u(k,t) &= -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3} u_m(k',t)u_n(k-k',t)dk' ik_m e_n \\
ik\cdot u(k,t) &= 0 \qquad\qquad (3.1) \\
u(k,0) &= u_0(k), \quad u_0\in\mathscr{S}, \quad ik\cdot u_0(k) = 0.
\end{aligned}
$$

3.3 Transformation over time

Consider the system (3.1). We zero-extend the field u in time to the real axis. The derivative of an extension is still calculated according to limitations due to the domain of definition $[0, \infty)$. In practice this means, that for the extension $\dot{u}$ the derivative at zero $\dot{u}(k, 0)$ is the right derivative at 0. We transform the system over time using Fourier transform

$$\mathscr{F}_t u(k, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(k, t) e^{-i\omega t} dt.$$

Define $f(k) = \frac{1}{2\pi} \mathscr{F} v_0(k)$. Assume $k \neq 0$. For the time derivative term we calculate

$$\begin{aligned}
\mathscr{F}_t \dot{u}(k, \omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \dot{u}(k, t) e^{-i\omega t} dt = \frac{1}{2\pi} \int_{0}^{\infty} \dot{u}(k, t) e^{-i\omega t} dt \\
&= \frac{1}{2\pi} \left[\Big/_{0}^{\infty} u(k, t) e^{-i\omega t} - \int_{0}^{\infty} u(k, t)(-i\omega) e^{-i\omega t} dt \right] \\
&= \frac{1}{2\pi} \left[0 - u(k, 0) + i\omega \int_{0}^{\infty} u(k, t) e^{-i\omega t} dt \right] \\
&= -\frac{1}{2\pi} u(k, 0) + i\omega \frac{1}{2\pi} \int_{0}^{\infty} u(k, t) e^{-i\omega t} dt \\
&= -\frac{1}{2\pi} \mathscr{F} v_0(k) + i\omega \frac{1}{2\pi} \int_{-\infty}^{\infty} u(k, t) e^{-i\omega t} dt \\
&= -f(k) + i\omega \mathscr{F}_t u(k, \omega).
\end{aligned}$$

Transformation of the laplacian term is elementary. Transformation of the convection term is more difficult. Note that projection $\hat{\Pi}^S$ is defined by dot product expression, that can be decomposed to sums of multiplication by components of s. Components of s are constant with respect to t. Fourier transform is linear and hence it passes also through sums. Transform of difference of identity and dot product part is also linear. Hence Fourier transform passes through $\hat{\Pi}^S$. In order to derive the result in a mathematically valid way, we assume $k \neq 0$, set $h = \mathscr{F}_t u$ and transform the dot product $k \cdot u(k, t)$ over time to obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} k \cdot u(k, t) e^{-i\omega t} dt &= \int_{-\infty}^{\infty} \sum_{n=1}^{3} k_n u(k, t)_n e^{-i\omega t} dt \\
&= \sum_{n=1}^{3} \int_{-\infty}^{\infty} k_n u(k, t)_n e^{-i\omega t} dt \\
&= \sum_{n=1}^{3} k_n \int_{-\infty}^{\infty} u(k, t)_n e^{-i\omega t} dt \\
&= \sum_{n=1}^{3} k_n \int_{-\infty}^{\infty} (u(k, t) e^{-i\omega t})_n dt \\
&= \sum_{n=1}^{3} k_n \left[\int_{-\infty}^{\infty} u(k, t) e^{-i\omega t} dt \right]_n \\
&= k \cdot \int_{-\infty}^{\infty} u(k, t) e^{-i\omega t} dt.
\end{aligned}$$

We now transform the field $\hat{\Pi}^I(k)u(k,t)$ over time to obtain

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\Pi}^I(k)u(k,t)e^{-i\omega t}dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}k\frac{1}{|k|^2}k\cdot u(k,t)e^{-i\omega t}dt \\
&= \frac{1}{2\pi}k\int_{-\infty}^{\infty}\frac{1}{|k|^2}k\cdot u(k,t)e^{-i\omega t}dt \\
&= \frac{1}{2\pi}k\frac{1}{|k|^2}\int_{-\infty}^{\infty}k\cdot u(k,t)e^{-i\omega t}dt \\
&= \frac{1}{2\pi}k\frac{1}{|k|^2}k\cdot\int_{-\infty}^{\infty}u(k,t)e^{-i\omega t}dt \\
&= k\frac{1}{|k|^2}k\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}u(k,t)e^{-i\omega t}dt \\
&= \hat{\Pi}^I(k)\frac{1}{2\pi}\int_{-\infty}^{\infty}u(k,t)e^{-i\omega t}dt.
\end{aligned}$$

We transform the field $\hat{\Pi}^S(k)u(k,t)$ over time to obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\Pi}^S(k)u(k,t)e^{-i\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}(u(k,t)-\hat{\Pi}^I(k)u(k,t))e^{-i\omega t}dt$$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}u(k,t)e^{-i\omega t}dt - \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\Pi}^I(k)u(k,t)e^{-i\omega t}dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}u(k,t)e^{-i\omega t}dt - \hat{\Pi}^I(k)\frac{1}{2\pi}\int_{-\infty}^{\infty}u(k,t)e^{-i\omega t}dt \\
&= \hat{\Pi}^S(k)\frac{1}{2\pi}\int_{-\infty}^{\infty}u(k,t)e^{-i\omega t}dt.
\end{aligned}$$

For vector fields u we have

$$\mathscr{F}_t u_j = (\mathscr{F}_t u)_j = h_j.$$

We obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}u_m(k',t)u_n(k-k',t)dk'ik_me_ne^{-i\omega t}dt \\
&= \hat{\Pi}^S(k)\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}u_m(k',t)u_n(k-k',t)dk'ik_me_ne^{-i\omega t}dt \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}u_m(k',t)u_n(k-k',t)dk'ik_me_ne^{-i\omega t}dt \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}u_m(k',t)u_n(k-k',t)dk'e^{-i\omega t}ik_me_ndt \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}u_m(k',t)u_n(k-k',t)dk'e^{-i\omega t}ik_mdte_n
\end{aligned}$$

$$
\begin{aligned}
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\frac{1}{2\pi}\int_{-\infty}^{\infty} ik_m \int_{\mathbb{R}^3} u_m(k',t)u_n(k-k',t)dk'e^{-i\omega t}dte_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3} ik_m\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} u_m(k',t)u_n(k-k',t)dk'e^{-i\omega t}dte_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} u_m(k',t)u_n(k-k',t)dk'e^{-i\omega t}dtik_me_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\frac{1}{2\pi}\int_{\mathbb{R}^3}\int_{-\infty}^{\infty} u_m(k',t)u_n(k-k',t)e^{-i\omega t}dtdk'ik_me_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\frac{1}{2\pi}\int_{-\infty}^{\infty} u_m(k',t)u_n(k-k',t)e^{-i\omega t}dtdk'ik_me_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{-\infty}^{\infty} \mathscr{F}_t u_m(k',\omega')\mathscr{F}_t u_n(k-k',\omega-\omega')d\omega'dk'ik_me_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \mathscr{F}_t u_m(k',\omega')\mathscr{F}_t u_n(k-k',\omega-\omega')d\omega'dk'ik_me_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3} \mathscr{F}_t u_m * \mathscr{F}_t u_n(k,\omega)ik_me_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3} h_m * h_n(k,\omega)ik_me_n,
\end{aligned}
$$

$$
-f(k) + i\omega h(k,\omega) + \nu|k|^2h(k,\omega) \quad = \quad -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3} h_m * h_n(k,\omega)ik_me_n.
$$

For notational convenience we write u instead of h. Hence we obtain

$$
(i\omega+\nu|k|^2)u(k,\omega) = -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3} u_m * u_n(k,\omega)ik_me_n + f(k). \tag{3.2}
$$

This form is called the subsidiary equation of Navier-Stokes equations. We calculate further

$$
\begin{aligned}
\sum_{n=1}^{3}\sum_{m=1}^{3} u_m * u_n(k,\omega)ik_me_n &= \sum_{n=1}^{3}\sum_{m=1}^{3}[u*u]_{n,m}(k,\omega)ik_me_n \\
&= \sum_{n=1}^{3}\sum_{m=1}^{3}[u*u(k,\omega)]_{n,m}[ik]_me_n \\
&= \sum_{n=1}^{3}[u*u(k,\omega)ik]_ne_n \\
&= u*u(k,\omega)ik
\end{aligned}
$$

and obtain the matrix formulation

$$
(i\omega+\nu|k|^2)u(k,\omega) = -\hat{\Pi}^S(k)\,u*u(k,\omega)\,ik + f(k) \tag{3.3}
$$

for the equation (3.2). Note that also $\hat{\Pi}^S(k)$ has matrix representation for every $k \neq 0$.

Chapter 4

Boundedness estimates for the subsidiary equation

The existence of solutions for many ordinary differential equations is proved using integral operator formulation and uniform convergence. In this chapter we introduce some equivalent formulations for equations (3.2) and (3.3). For each formulation we rewrite the equation in the form

$$u = Tu + g$$

and estimate the operator T using a suitable norm.

4.1 Transfer function estimate

We now have the subsidiary equation (3.2) for Navier-Stokes equations (2.3) and we proceed according to the operational calculus method in [6], that is, we divide both sides of the equation by $i\omega + \nu|k|^2$ to obtain

$$u(k,\omega) \quad = \quad -\frac{1}{i\omega + \nu|k|^2}\hat{\Pi}^S(k)\, u * u(k,\omega)\, ik + \frac{f(k)}{i\omega + \nu|k|^2}. \tag{4.1}$$

We calculate

$$\begin{aligned}
q(k,\omega)\hat{\Pi}^S(k)u * u(k,\omega)ik &= q(k,\omega)\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3} u_m * u_n(k,\omega)ik_m e_n \\
&= \hat{\Pi}^S(k)q(k,\omega)\sum_{n=1}^{3}\sum_{m=1}^{3} u_m * u_n(k,\omega)ik_m e_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\left[q(k,\omega)\sum_{m=1}^{3} u_m * u_n(k,\omega)ik_m\right] e_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3} q(k,\omega)u_m * u_n(k,\omega)ik_m e_n \\
&= \hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3} u_m * u_n(k,\omega)q(k,\omega)ik_m e_n
\end{aligned}$$

$$
\begin{aligned}
&= \hat{\Pi}^S(k) \sum_{n=1}^{3} \sum_{m=1}^{3} u_m * u_n(k,\omega)(q(k,\omega)ik)_m e_n \\
&= \hat{\Pi}^S(k)\, u * u(k,\omega)\, q(k,\omega) ik.
\end{aligned}
$$

The result is nontrivial and allows to analyze the convolution factor $u * u$ and the transfer function factor $\frac{ik}{i\omega+\nu|k|^2}$ separately. We calculate

$$
\begin{aligned}
\frac{1}{i\omega+\nu|k|^2}\hat{\Pi}^S(k)\, u * u(k,\omega)\, ik &= \hat{\Pi}^S(k)\, u * u(k,\omega)\, \frac{1}{i\omega+\nu|k|^2} ik \\
&= \hat{\Pi}^S(k)\, u * u(k,\omega)\, \frac{ik}{i\omega+\nu|k|^2}
\end{aligned}
$$

and obtain the formulation

$$
u(k,\omega) = -\hat{\Pi}^S(k)\, u * u(k,\omega)\, \frac{ik}{i\omega+\nu|k|^2} + \frac{f(k)}{i\omega+\nu|k|^2} \tag{4.2}
$$

for the equation (4.1). We define

$$
\begin{aligned}
Tu(k,\omega) &= -\hat{\Pi}^S(k)\, u * u(k,\omega)\, \frac{ik}{i\omega+\nu|k|^2}, \\
g(k,\omega) &= \frac{f(k)}{i\omega+\nu|k|^2}
\end{aligned} \tag{4.3}
$$

and write (4.2) in the form

$$
\begin{aligned}
u(k,\omega) &= Tu(k,\omega) + g(k,\omega) = (Tu+g)(k,\omega) \\
u &= Tu + g.
\end{aligned} \tag{4.4}
$$

The next phase in our analysis is to find a norm for which the operator T is contractive. Consider an initial value problem and its solution x, $x(t) = 0$ for $t < 0$. We know that Volterra operator

$$
Vx(t) = \int_0^t x(t')dt' \tag{4.5}
$$

is contractive in exponentially weighted supremum norm

$$
||x|| = \sup_{t\geq 0} |x(t)e^{-\alpha t}|. \tag{4.6}
$$

with some $\alpha \in \mathbb{R}^+$. To show this, we calculate

$$
\begin{aligned}
||Vx|| &= \sup_{t\geq 0} |Vx(t)e^{-\alpha t}| = \sup_{t\geq 0} |Vx(t)||e^{-\alpha t}| = \sup_{t\geq 0} \left| \int_0^t x(t')dt' \right| e^{-\alpha t} \\
&\leq \sup_{t\geq 0} \int_0^t |x(t')|dt' e^{-\alpha t} = \sup_{t\geq 0} \int_0^t |x(t')|e^{-\alpha t'} e^{\alpha t'} dt' e^{-\alpha t} \\
&= \sup_{t\geq 0} \int_0^t |x(t')e^{-\alpha t'}| e^{\alpha t'} dt' e^{-\alpha t} \leq \sup_{t\geq 0} \int_0^t ||x|| e^{\alpha t'} dt' e^{-\alpha t} \\
&= \sup_{t\geq 0} ||x|| \int_0^t e^{\alpha t'} dt' e^{-\alpha t} = ||x|| \sup_{t\geq 0} \int_0^t e^{\alpha t'} dt' e^{-\alpha t}
\end{aligned}
$$

$$
\begin{aligned}
&= ||x||\sup_{t\geq 0}\Big/_0^t \frac{1}{\alpha}e^{\alpha t'}e^{-\alpha t} = ||x||\sup_{t\geq 0}\frac{1}{\alpha}(e^{\alpha t}-1)e^{-\alpha t}\\
&= ||x||\sup_{t\geq 0}\frac{1}{\alpha}(1-e^{-\alpha t}) \leq ||x||\sup_{t\geq 0}\frac{1}{\alpha} = ||x||\frac{1}{\alpha}\\
&= \frac{1}{\alpha}||x||\\
||V|| &= \sup_{||x||=1}||Vx|| \leq \sup_{||x||=1}\frac{1}{\alpha}||x|| = \frac{1}{\alpha}\sup_{||x||=1}||x||\\
&= \frac{1}{\alpha}\sup_{||x||=1} 1 = \frac{1}{\alpha}1 = \frac{1}{\alpha}.
\end{aligned}
$$

We notice that exponential weighting by $e^{-\alpha t}$ is transformed into the form of an imaginary translation. To show this we calculate

$$
\begin{aligned}
\mathscr{F}f(\omega - i\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i(\omega-i\alpha)t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t-\alpha t}dt\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}e^{-\alpha t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-\alpha t}e^{-i\omega t}dt.
\end{aligned}
$$

Hence for every $a \in -\overline{\Pi^-} = \overline{\Pi^+}$ we define translation

$$\check{\tau}_a:\ Z^3 \to Z^3,\quad \tau_a u(k,\omega) = u(k,\omega - a).$$

Translation $\check{\tau}_a$ is linear. The proof is similar to the linearity proof of τ_α. We now define function

$$|||u|||_\alpha = ||\check{\tau}_{i\alpha^2}u||,\quad \alpha > 0. \tag{4.7}$$

To show that this is a norm, we establish boundedness and continuity of $u(k,\omega)$, $k \neq 0$, to the boundary $\mathbb{R}$ with respect to the variable ω. We assume that $\mathscr{F}_x v(k,t)$ is bounded and absolutely integrable for every $k \neq 0$. Fix $k \neq 0$. Fix $\epsilon > 0$. Fix $M_1 > 0$ such that

$$\int_{M_1}^{\infty} |f(k,t)|dt < \frac{\epsilon}{2}.$$

Fix $\delta = -\frac{1}{M}\ln\left(1 - \frac{\epsilon}{2||f||_\infty M}\right)$. Take any $\alpha > 0$, $|\alpha - 0| < \delta$. We calculate

$$
\begin{aligned}
&\left|\int_0^\infty f(k,t)e^{-i(\omega-i\alpha)t}dt - \int_0^\infty f(k,t)e^{-i\omega t}dt\right|\\
&= \left|\int_0^\infty (e^{-\alpha t}-1)f(k,t)e^{-i\omega t}dt\right|\\
&= \left|\int_0^M (e^{-\alpha t}-1)f(k,t)e^{-i\omega t}dt - \int_M^\infty (e^{-\alpha t}-1)f(k,t)e^{-i\omega t}dt\right|\\
&\leq \left|\int_0^M (e^{-\alpha t}-1)f(k,t)e^{-i\omega t}dt\right| + \left|\int_M^\infty (e^{-\alpha t}-1)f(k,t)e^{-i\omega t}dt\right|\\
&\leq \int_0^M |e^{-\alpha t}-1||f(t)|dt + \int_M^\infty |e^{-\alpha t}-1||f(t)|dt\\
&\leq \int_0^M (1-e^{-\alpha M})|f(t)|dt + \int_M^\infty |f(t)|dt
\end{aligned}
$$

$$= \quad ||f||_\infty M(1-e^{-\alpha M}) + \int_M^\infty |f(t)|dt < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $u(\omega, k)$ is continuous to the boundary $\mathbb{R}$ with respect to the variable ω. Boundedness is clear for $\mathrm{Im}(\omega) \leq 0$.

Assume that $f : \mathbb{R} \to \mathbb{C}$ and $f \in L^2([0,\infty))$. Then $\mathscr{F} f$ has complex derivative in the lower complex plane. This is shown in [11] in the introduction of Section 19. Hence also the space transformed vector field component $f(k,\omega)$, $k \neq 0$, is analytic on the lower complex plane. Continuity to the boundary $\mathbb{R}$ was established already.

We calculate

$$|||x|||_\alpha \quad = \quad ||\check{\tau}_{i\alpha^2} x|| \geq 0.$$

Hence $|||\cdot|||_\alpha$ is positive. Linearity of $\check{\tau}_a$ shows $\check{\tau}_a 0 = 0$. We calculate

$$|||0|||_\alpha = ||\tau_{i\alpha^2} 0|| = ||0|| = 0.$$

Assume $|||x|||_\alpha = 0$. Fix $k \neq 0$. Components $x_n(k, \omega - i\alpha^2)$ over ω are analytic and hence continuous. This implies that they all are zero for every $\omega \in \mathbb{R}$. Assume any component $x_n(k,\omega), k \neq 0$, that is complex analytic on the lower complex plane and has the value zero on the line $\mathrm{Im}(\omega) = -\alpha^2$. It has complex Taylor series representation and derivatives 0 on the line $\mathrm{Im}(\omega) = -\alpha^2$. Note that complex derivative can be calculated on the real line.

Fix $\omega = 0$. The function x_n is zero in the neighborhood $B(\omega - i\alpha^2, \epsilon)$ of the point $\omega - i\alpha^2$ by Taylor's theorem. We recall that region is an open, connected set. Hence the lower complex plane is a region. The function 0 defined on the lower complex plane is differentiable and hence analytic. It is an analytic continuation of x_n from $B(\omega - i\alpha^2, \epsilon)$ to the lower complex plane. Theorem 10.18 in [11] shows uniqueness of analytic continuation to the lower complex plane. Hence $x_n = 0$ on the lower complex plane. Continuity of x_n shows that $x_n = 0$ also on $\mathbb{R}$. This implies $x_n = 0$ for every n. Hence $x = 0$ and we have

$$x = 0 \quad \Leftrightarrow \quad |||x|||_\alpha = 0.$$

We calculate

$$|||ax|||_\alpha \quad = \quad ||\check{\tau}_{i\alpha^2}(ax)|| = ||a\check{\tau}_{i\alpha^2} x|| = |a| ||\check{\tau}_{i\alpha^2} x|| = |a| |||x|||_\alpha.$$

Hence $|||x|||_\alpha$ is homogenous. We calculate

$$\begin{aligned} |||x+y|||_\alpha \quad &= \quad ||\check{\tau}_{i\alpha^2}(x+y)|| = ||\check{\tau}_{i\alpha^2} x + \check{\tau}_{i\alpha^2} y|| \leq ||\check{\tau}_{i\alpha^2} x|| + ||\check{\tau}_{i\alpha^2} y|| \\ &= \quad |||x|||_\alpha + |||y|||_\alpha. \end{aligned}$$

Hence $|||\cdot|||_\alpha$ satisfies triangle inequality. This shows that $|||\cdot|||_\alpha$ is a norm.

Definition 4.1. Assume that $x \in \mathbb{C}^3$ and $T \in L(\mathbb{C}^3)$. We define

$$\begin{aligned} |x| \quad &= \quad \sqrt{\sum_{k=1}^3 |x_k|^2} \\ |T| \quad &= \quad \sup_{|x|=1} |Tx|. \end{aligned}$$

Lemma 4.2. *Assume sufficiently smooth functions $f \in W$ and $g \in W$ such that $||f||_1 \in \mathbb{R}$ and $||g|| \in \mathbb{R}$. Then we have the estimate*

$$||f * g|| \leq 9\,||f||_1\,||g||.$$

Proof: We calculate

$$\begin{aligned}
||f*g|| &= \sup_{k\in S,\omega\in\mathbb{R}} |f*g(k,\omega)| = \sup_{k\in S,\omega\in\mathbb{R}}\sup_{|x|=1} |f*g(k,\omega)x| \\
&= \sup_{k\in S,\omega\in\mathbb{R}}\sup_{|x|=1} \left|\sum_{k=1}^{3}\sum_{j=1}^{3}[f*g]_{k,j}(k,\omega)x_j e_k\right| \\
&= \sup_{k\in S,\omega\in\mathbb{R}}\sup_{|x|=1} \left|\sum_{k=1}^{3}\sum_{j=1}^{3} f_j * g_k(k,\omega)x_j e_k\right| \\
&\leq \sup_{k\in S,\omega\in\mathbb{R}}\sup_{|x|=1} \sum_{k=1}^{3}\left|\sum_{j=1}^{3} f_j * g_k(k,\omega)x_j\right|\left|e_k\right| \\
&= \sup_{k\in S,\omega\in\mathbb{R}}\sup_{|x|=1} \sum_{k=1}^{3}\left|\sum_{j=1}^{3} f_j * g_k(k,\omega)x_j\right| \\
&\leq \sup_{k\in S,\omega\in\mathbb{R}}\sup_{|x|=1} \sum_{k=1}^{3}\sum_{j=1}^{3} |f_j * g_k(k,\omega)||x_j| \\
&\leq \sup_{k\in S,\omega\in\mathbb{R}}\sup_{|x|=1} \sum_{k=1}^{3}\sum_{j=1}^{3} |f_j * g_k(k,\omega)||x| \\
&\leq \sup_{k\in S,\omega\in\mathbb{R}} \sum_{k=1}^{3}\sum_{j=1}^{3} \left|\int_S\int_{\mathbb{R}} f_j(k',\omega')g_k(k-k',\omega-\omega')dk\,d\omega\right| \\
&\leq \sup_{k\in S,\omega\in\mathbb{R}} \sum_{k=1}^{3}\sum_{j=1}^{3} \int_S\int_{\mathbb{R}} |f_j(k',\omega')||g_k(k-k',\omega-\omega')|dk\,d\omega \\
&\leq \sup_{k\in S,\omega\in\mathbb{R}} \sum_{k=1}^{3}\sum_{j=1}^{3} \int_S\int_{\mathbb{R}} |f(k',\omega')_j||g(k-k',\omega-\omega')_k|dk\,d\omega \\
&\leq \sup_{k\in S,\omega\in\mathbb{R}} \sum_{k=1}^{3}\sum_{j=1}^{3} \int_S\int_{\mathbb{R}} |f(k',\omega')||g(k-k',\omega-\omega')|dk\,d\omega \\
&= \sup_{k\in S,\omega\in\mathbb{R}} 9\int_S\int_{\mathbb{R}} |f(k',\omega')||g(k-k',\omega-\omega')|dkd\omega \\
&\leq \sup_{k\in S,\omega\in\mathbb{R}} 9\int_S\int_{\mathbb{R}} |f(k',\omega')|||g||dk\,d\omega \\
&\leq \sup_{k\in S,\omega\in\mathbb{R}} 9\,||g||\int_S\int_{\mathbb{R}} |f(k',\omega')|dk\,d\omega \\
&\leq \sup_{k\in S,\omega\in\mathbb{R}} 9||g||||f||_1 = \sup_{k\in S,\omega\in\mathbb{R}} 9||f||_1||g|| = 9\,||f||_1||g||
\end{aligned}$$

to obtain the claim. □

Lemma 4.3. *Assume $\alpha \in \mathbb{R}^+$ and sufficiently smooth functions $f \in Z$ and $g \in Z$ such that convolution $f * g$ exists. Then we have the equation*

$$\check{\tau}_{i\alpha^2}(f * g) = f * \check{\tau}_{i\alpha^2} g.$$

Proof: Fix any indices $i, j \in \{1, 2, 3\}$. We calculate

$$\begin{aligned}
[\check{\tau}_{i\alpha^2}(f * g)(k, \omega)]_{i,j} &= [f * g(k, \omega - i\alpha^2)]_{i,j} = [f * g]_{i,j}(k, \omega - i\alpha^2) \\
&= f_j * g_i(k, \omega - i\alpha^2) \\
&= \int_S \int_{\mathbb{R}} f_j(k', \omega') g_i(k - k', \omega - i\alpha^2 - \omega') d\omega\, dk \\
&= \int_S \int_{\mathbb{R}} f_j(k', \omega') g_i(k - k', \omega - \omega' - i\alpha^2) d\omega\, dk \\
&= \int_S \int_{\mathbb{R}} f_j(k', \omega') g(k - k', \omega - \omega' - i\alpha^2)_i d\omega\, dk \\
&= \int_S \int_{\mathbb{R}} f_j(k', \omega') \check{\tau}_{i\alpha^2} g(k - k', \omega - \omega')_i d\omega\, dk \\
&= \int_S \int_{\mathbb{R}} f_j(k', \omega') (\check{\tau}_{i\alpha^2} g)_i(k - k', \omega - \omega') d\omega\, dk \\
&= f_j * (\check{\tau}_{i\alpha^2} g)_i(k, \omega) = [f * \check{\tau}_{i\alpha^2} g]_{i,j}(k, \omega) \\
&= [f * \check{\tau}_{i\alpha^2} g(k, \omega)]_{i,j}
\end{aligned}$$

$$\begin{aligned}
\check{\tau}_{i\alpha^2}(f * g)(k, \omega) &= f * \check{\tau}_{i\alpha^2} g(k, \omega) \\
\check{\tau}_{i\alpha^2}(f * g) &= f * \check{\tau}_{i\alpha^2} g.
\end{aligned}$$

and obtain the claim. □

Lemma 4.4. *We have the inequality*

$$\frac{|k|}{\alpha^2 + \nu|k|^2} \leq \frac{1}{\alpha\sqrt{\nu}}.$$

Proof: We estimate

$$\begin{aligned}
\frac{|k|}{\alpha^2 + \nu|k|^2} &= \frac{1}{\sqrt{\alpha^2 + \nu|k|^2}} \frac{\sqrt{\nu|k|^2}}{\sqrt{\alpha^2 + \nu|k|^2}} \frac{1}{\sqrt{\nu}} \leq \frac{1}{\sqrt{\alpha^2}} \frac{\sqrt{\alpha^2 + \nu|k|^2}}{\sqrt{\alpha^2 + \nu|k|^2}} \frac{1}{\sqrt{\nu}} \\
&= \frac{1}{\alpha\sqrt{\nu}}
\end{aligned}$$

and obtain the claim. □

We have collected sufficient knowledge to estimate Tu in α-norm. We calculate

$$\begin{aligned}
|\check{\tau}_{i\alpha^2} Tu(k, \omega)| &= |Tu(k, \omega - i\alpha^2)| \\
&= \left| -\hat{\Pi}^S(k)\, u * u(k, \omega - i\alpha^2) \frac{ik}{i(\omega - i\alpha^2) + \nu|k|^2} \right| \\
&= \left| -\hat{\Pi}^S(k)\, \check{\tau}_{i\alpha^2} u * u(k, \omega) \frac{ik}{i\omega - ii\alpha^2 + \nu|k|^2} \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| -\hat{\Pi}^S(k)\, u * \check{\tau}_{i\alpha^2} u(k,\omega)\, \frac{ik}{i\omega + \alpha^2 + \nu|k|^2} \right| \\
&\le \left| -\hat{\Pi}^S(k) \right| \left| u * \check{\tau}_{i\alpha^2} u(k,\omega)\, \frac{ik}{i\omega + \alpha^2 + \nu|k|^2} \right| \\
&\le \left| \hat{\Pi}^S(k) \right| \left| u * \check{\tau}_{i\alpha^2} u(k,\omega) \right| \left| \frac{ik}{i\omega + \alpha^2 + \nu|k|^2} \right| \\
&\le 2\, ||u * \check{\tau}_{i\alpha^2} u||\, \frac{|k|}{|i\omega + \alpha^2 + \nu|k|^2|} \\
&\le 2 \cdot 9\, ||u||_1 ||\check{\tau}_{i\alpha^2} u||\, \frac{|k|}{|\alpha^2 + \nu|k|^2|} \\
&= 18\, ||u||_1 ||\check{\tau}_{i\alpha^2} u||\, \frac{|k|}{\alpha^2 + \nu|k|^2} \\
&\le 18 ||u||_1 |||u|||_\alpha \frac{1}{\alpha\sqrt{\nu}} = \frac{18}{\alpha\sqrt{\nu}} ||u||_1 |||u|||_\alpha
\end{aligned}
$$

$$
\begin{aligned}
|\check{\tau}_{i\alpha^2} g(k,\omega)| &= |g(k, \omega - i\alpha^2)| = \left| \frac{f(k)}{i(\omega - i\alpha^2) + \nu|k|^2} \right| \\
&= \left| \frac{f(k)}{i\omega - ii\alpha^2 + \nu|k|^2} \right| = \left| \frac{f(k)}{i\omega + \alpha^2 + \nu|k|^2} \right| \\
&= \frac{|f(k)|}{|i\omega + \alpha^2 + \nu|k|^2|} \le \frac{|f(k)|}{|\alpha^2 + \nu|k|^2|} \\
&\le \frac{|f(k)|}{|\alpha^2|} \le \frac{1}{\alpha^2} ||f||
\end{aligned}
$$

$$
\begin{aligned}
|||Tu|||_\alpha &= ||\check{\tau}_{i\alpha^2} Tu|| = \sup_{k\in S, \omega\in\mathbb{R}} |\check{\tau}_{i\alpha^2} Tu(k,\omega)| \\
&\le \sup_{k\in S, \omega\in\mathbb{R}} \frac{18}{\alpha\sqrt{\nu}} ||u||_1 |||u|||_\alpha = \frac{18}{\alpha\sqrt{\nu}} ||u||_1 |||u|||_\alpha
\end{aligned}
$$

$$
|||g|||_\alpha = ||\check{\tau}_{i\alpha^2} g|| = \sup_{k\in S, \omega\in\mathbb{R}} |\check{\tau}_{i\alpha^2} g(k,\omega)| \le \sup_{k\in S, \omega\in\mathbb{R}} \frac{1}{\alpha^2} ||f|| = \frac{1}{\alpha^2} ||f||.
$$

The function g is, unfortunately, not in L^1, which advices to search for more sophisticated estimates.

4.2 Estimate for L^2 - L^1 norm

$L^2 - L^1$ method uses the equation (4.2) and definitions (4.3) leading to the formulation (4.4). We apply a set of norms indexed by real parameter $\alpha > 0$. The definition of the set of norms differs from the definition applied in Section 4.1. We define

$$
|||u|||_\alpha = \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\check{\tau}_{i\alpha^2} u(k,\omega)|^2 d\omega \right]^{\frac{1}{2}} dk. \tag{4.8}
$$

We will show that $|||u|||_\alpha$ is a norm. We calculate

$$|||u|||_\alpha = \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\acute{\tau}_{i\alpha^2} u(k,\omega)|^2 d\omega^{\frac{1}{2}} dk \geq \alpha \int_{\mathbb{R}^3} \int_{\mathbb{R}} 0 d\omega \right]^{\frac{1}{2}} dk = 0.$$

This shows that $|||u|||_\alpha$ is nonnegative. We calculate

$$\begin{aligned} |||0|||_\alpha &= \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\acute{\tau}_{i\alpha^2} 0(k,\omega)|^2 d\omega^{\frac{1}{2}} dk = \alpha \int_{\mathbb{R}^3} \int_{\mathbb{R}} |0(k,\omega - i\alpha^2)|^2 d\omega \right]^{\frac{1}{2}} dk \\ &= \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |0|^2 d\omega \right]^{\frac{1}{2}} dk = \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} 0 d\omega \right]^{\frac{1}{2}} dk = 0. \end{aligned}$$

Assume that $|||u|||_\alpha = 0$. Countinuity of $u(k,\omega)$ over ω was shown in Section 4.1 for every $k \neq 0$. In addition we assume that $u(k,\omega)$ is continuous on Z. Continuity of u on Z implies $\acute{\tau}_{i\alpha^2} u(k,\omega) = 0$ for every $(k,\omega) \in S \times \mathbb{R}$ and hence on the line $\mathrm{Im}(\omega) = -\alpha^2$. Analyticity and continuity to the boundary $\mathbb{R}$ of u implies $u = 0$ (see respective proof in Section 4.1). We obtain

$$|||u|||_\alpha = 0 \Leftrightarrow u = 0.$$

We calculate applying properties of the 2-norm to obtain

$$\begin{aligned} |||\beta u|||_\alpha &= \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\acute{\tau}_{i\alpha^2} \beta u(k,\omega)|^2 d\omega \right]^{\frac{1}{2}} dk \\ &= \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\beta \acute{\tau}_{i\alpha^2} u(k,\omega)|^2 d\omega \right]^{\frac{1}{2}} dk \\ &= \alpha \int_{\mathbb{R}^3} |\beta| \left[\int_{\mathbb{R}} |\beta \acute{\tau}_{i\alpha^2} u(k,\omega)|^2 d\omega \right]^{\frac{1}{2}} dk \\ &= \alpha|\beta| \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\acute{\tau}_{i\alpha^2} u(k,\omega)|^2 d\omega \right]^{\frac{1}{2}} dk \\ &= |\beta|\alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\acute{\tau}_{i\alpha^2} u(k,\omega)|^2 d\omega \right]^{\frac{1}{2}} dk \\ &= |\beta| |||u|||_\alpha. \end{aligned}$$

This shows homogeneity of $|||u|||_\alpha$. We calculate

$$\begin{aligned} |||u_1 + u_2|||_\alpha &= \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\acute{\tau}_{i\alpha^2} (u_1 + u_2)(k,\omega)|^2 d\omega \right]^{\frac{1}{2}} dk \\ &= \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |(\acute{\tau}_{i\alpha^2} u_1 + \acute{\tau}_{i\alpha^2} u_2)(k,\omega)|^2 d\omega \right]^{\frac{1}{2}} dk \\ &= \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\acute{\tau}_{i\alpha^2} u_1(k,\omega) + \acute{\tau}_{i\alpha^2} u_2(k,\omega)|^2 d\omega \right]^{\frac{1}{2}} dk \\ &\leq \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\acute{\tau}_{i\alpha^2} u_1(k,\omega)|^2 d\omega^{\frac{1}{2}} + \int_{\mathbb{R}} |\acute{\tau}_{i\alpha^2} u_2(k,\omega)|^2 d\omega \right]^{\frac{1}{2}} dk \end{aligned}$$

$$
\begin{aligned}
&= \alpha\Big[\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}|\check{\tau}_{i\alpha^2}u_1(k,\omega)|^2 d\omega\Big]^{\frac{1}{2}}dk + \int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}|\check{\tau}_{i\alpha^2}u_2(k,\omega)|^2 d\omega\Big]^{\frac{1}{2}}dk\Big] \\
&= \alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}|\check{\tau}_{i\alpha^2}u_1(k,\omega)|^2 d\omega\Big]^{\frac{1}{2}}dk + \alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}|\check{\tau}_{i\alpha^2}u_2(k,\omega)|^2 d\omega\Big]^{\frac{1}{2}}dk \\
&= |||u_1|||_\alpha + |||u_2|||_\alpha.
\end{aligned}
$$

This shows that $|||\cdot|||_\alpha$ satisfies the triangle inequality and completes the proof. We estimate the operator T and obtain

$$
\begin{aligned}
|||Tu|||_\alpha &= \alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}|\check{\tau}_{i\alpha^2}Tu(k,\omega)|^2 d\omega\Big]^{\frac{1}{2}}dk \\
&= \alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}|Tu(k,\omega-i\alpha^2)|^2 d\omega\Big]^{\frac{1}{2}}dk \\
&= \alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}\Big|-\hat{\Pi}^S(k)\,u*u(k,\omega-i\alpha^2) \\
&\qquad \frac{ik}{i(\omega-i\alpha^2)+\nu|k|^2}\Big|^2 d\omega\Big]^{\frac{1}{2}}dk \\
&= \alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}\Big|-\hat{\Pi}^S(k)\,u*u(k,\omega-i\alpha^2)\frac{ik}{i\omega-ii\alpha^2+\nu|k|^2}\Big|^2 d\omega\Big]^{\frac{1}{2}}dk \\
&= \alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}\Big|-\hat{\Pi}^S(k)\,u*u(k,\omega-i\alpha^2)\frac{ik}{i\omega+\alpha^2+\nu|k|^2}\Big|^2 d\omega\Big]^{\frac{1}{2}}dk \\
&\le \alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}\Big|\hat{\Pi}^S(k)\Big|^2\Big|u*u(k,\omega-i\alpha^2)\frac{ik}{i\omega+\alpha^2+\nu|k|^2}\Big|^2 d\omega\Big]^{\frac{1}{2}}dk \\
&\le \alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}4\Big|\check{\tau}_{i\alpha^2}(u*u)(k,\omega)\frac{ik}{i\omega+\alpha^2+\nu|k|^2}\Big|^2 d\omega\Big]^{\frac{1}{2}}dk \\
&= 2\alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}\Big|u*\check{\tau}_{i\alpha^2}u(k,\omega)ik\frac{1}{i\omega+\alpha^2+\nu|k|^2}\Big|^2 d\omega\Big]^{\frac{1}{2}}dk \\
&= 2\alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}\Big|\sum_{n=1}^{3}\sum_{m=1}^{3}[u*\check{\tau}_{i\alpha^2}u(k,\omega)]_{n,m} \\
&\qquad ik_m e_n\frac{1}{i\omega+\alpha^2+\nu|k|^2}\Big|^2 d\omega\Big]^{\frac{1}{2}}dk \\
&= 2\alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}\Big|\sum_{n=1}^{3}\sum_{m=1}^{3}[u*\check{\tau}_{i\alpha^2}u]_{n,m}(k,\omega) \\
&\qquad ik_m e_n\frac{1}{i\omega+\alpha^2+\nu|k|^2}\Big|^2 d\omega\Big]^{\frac{1}{2}}dk \\
&= 2\alpha\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}}\Big|\sum_{n=1}^{3}\sum_{m=1}^{3}u_m*[\check{\tau}_{i\alpha^2}u]_n(k,\omega) \\
&\qquad ik_m e_n\frac{1}{i\omega+\alpha^2+\nu|k|^2}\Big|^2 d\omega\Big]^{\frac{1}{2}}dk
\end{aligned}
$$

$$
\begin{aligned}
&= 2\alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left| \sum_{n=1}^{3} \sum_{m=1}^{3} \int_{\mathbb{R}^3} \int_{\mathbb{R}} u_m(k',\omega')[\check{\tau}_{i\alpha^2} u]_n(k-k',\omega-\omega') \right.\right. \\
&\quad \left.\left. d\omega' dk' i k_m e_n \frac{1}{i\omega+\alpha^2+\nu|k|^2} \right|^2 d\omega \right]^{\frac{1}{2}} dk \\
&= 2\alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left[\left| \sum_{n=1}^{3} \sum_{m=1}^{3} \int_{\mathbb{R}^3} \int_{\mathbb{R}} u_m(k',\omega')[\check{\tau}_{i\alpha^2} u]_n(k-k',\omega-\omega') \right.\right.\right. \\
&\quad \left.\left.\left. d\omega' dk' i k_m e_n \right| \left| \frac{1}{i\omega+\alpha^2+\nu|k|^2} \right| \right]^2 d\omega \right]^{\frac{1}{2}} dk \\
&\le 2\alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left[\sum_{n=1}^{3} \left| \sum_{m=1}^{3} \int_{\mathbb{R}^3} \int_{\mathbb{R}} u_m(k',\omega')[\check{\tau}_{i\alpha^2} u]_n(k-k',\omega-\omega') \right.\right.\right. \\
&\quad \left.\left.\left. d\omega' dk' i k_m \right| \left| e_n \right| \left| \frac{1}{i\omega+\alpha^2+\nu|k|^2} \right| \right]^2 d\omega \right]^{\frac{1}{2}} dk \\
&\le 2\alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left[\sum_{n=1}^{3} \sum_{m=1}^{3} \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}} u_m(k',\omega')[\check{\tau}_{i\alpha^2} u]_n(k-k',\omega-\omega') \right.\right.\right. \\
&\quad \left.\left.\left. d\omega' dk' \right| \left| i \right| \left| k_m \right| \left| \frac{1}{i\omega+\alpha^2+\nu|k|^2} \right| \right]^2 d\omega \right]^{\frac{1}{2}} dk \\
&\le 2\alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left[\sum_{n=1}^{3} \sum_{m=1}^{3} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |u_m(k',\omega')| |[\check{\tau}_{i\alpha^2} u]_n(k-k',\omega-\omega')| \right.\right. \\
&\quad \left.\left. d\omega' dk' |k| \frac{1}{|i\omega+\alpha^2+\nu|k|^2|} \right]^2 d\omega \right]^{\frac{1}{2}} dk \\
&\le 2\alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left[\sum_{n=1}^{3} \sum_{m=1}^{3} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |u(k',\omega')_m| |\check{\tau}_{i\alpha^2} u(k-k',\omega-\omega')_n| d\omega' \right.\right. \\
&\quad \left.\left. dk' \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|} \right]^2 d\omega \right]^{\frac{1}{2}} dk \\
&\le 2\alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left[\sum_{n=1}^{3} \sum_{m=1}^{3} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |u(k',\omega')| |\check{\tau}_{i\alpha^2} u(k-k',\omega-\omega')| \right.\right. \\
&\quad \left.\left. d\omega' dk' \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|} \right]^2 d\omega \right]^{\frac{1}{2}} dk \\
&= 2\alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left[9 \int_{\mathbb{R}^3} \int_{\mathbb{R}} |u(k',\omega')| |\check{\tau}_{i\alpha^2} u(k-k',\omega-\omega')| d\omega' dk' \right.\right. \\
&\quad \left.\left. \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|} \right]^2 d\omega \right]^{\frac{1}{2}} dk \\
&\le 2\alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left[9 \int_{\mathbb{R}^3} \int_{\mathbb{R}} |u(k',\omega')|^2 d\omega'^{\frac{1}{2}} \int_{\mathbb{R}} |\check{\tau}_{i\alpha^2} u(k-k',\omega-\omega')|^2 d\omega'^{\frac{1}{2}} dk' \right.\right. \\
&\quad \left.\left. \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|} \right]^2 d\omega \right]^{\frac{1}{2}} dk
\end{aligned}
$$

$$
\begin{aligned}
&= 2\alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left[9 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |u(k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} |\grave{\tau}_{i\alpha^2} u(k-k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk' \right.\right. \\
&\quad \left.\left. \frac{|k|}{|i\omega + \alpha^2 + \nu|k|^2|} \right]^2 d\omega \right]^{\frac{1}{2}} dk \\
&\leq 2\alpha \int_{\mathbb{R}^3} \int_{\mathbb{R}} \left| 9 \int_{\mathbb{R}^3} \int_{\mathbb{R}} |u(k',\omega')|^2 d\omega'^{\frac{1}{2}} \int_{\mathbb{R}} |\grave{\tau}_{i\alpha^2} u(k-k',\omega')|^2 d\omega'^{\frac{1}{2}} dk' \right. \\
&\quad \left. \frac{|k|}{|i\omega + \alpha^2 + \nu|k|^2|} \right|^2 d\omega^{\frac{1}{2}} dk \\
&= 2\alpha \int_{\mathbb{R}^3} 9 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |u(k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} |\grave{\tau}_{i\alpha^2} u(k-k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&\quad \left[\int_{\mathbb{R}} \left| \frac{|k|}{|i\omega + \alpha^2 + \nu|k|^2|} \right|^2 d\omega \right]^{\frac{1}{2}} dk \\
&= 2\alpha \int_{\mathbb{R}^3} 9 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |u(k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} |\grave{\tau}_{i\alpha^2} u(k-k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&\quad \left[\int_{\mathbb{R}} \frac{|k|^2}{\omega^2 + (\alpha^2 + \nu|k|^2)^2} d\omega \right]^{\frac{1}{2}} dk \\
&= 2\alpha \int_{\mathbb{R}^3} 9 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |u(k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} |\grave{\tau}_{i\alpha^2} u(k-k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&\quad \left[\frac{|k|^2}{\alpha^2 + \nu|k|^2} \int_{\mathbb{R}} \frac{1}{\omega^2 + 1} d\omega \right]^{\frac{1}{2}} dk \\
&= 2\alpha \int_{\mathbb{R}^3} 9 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |u(k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} |\grave{\tau}_{i\alpha^2} u(k-k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&\quad \left[\frac{\nu|k|^2}{\alpha^2 + \nu|k|^2} \frac{\pi}{\nu} \right]^{\frac{1}{2}} dk \\
&\leq \frac{18\alpha\pi^{\frac{1}{2}}}{\nu^{\frac{1}{2}}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |u(k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} |\grave{\tau}_{i\alpha^2} u(k-k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk' dk \\
&= \frac{18\alpha\pi^{\frac{1}{2}}}{\nu^{\frac{1}{2}}} \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |u(k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\grave{\tau}_{i\alpha^2} u(k-k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk dk' \\
&= \frac{18\alpha\pi}{\nu^{\frac{1}{2}}} \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |u(k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\grave{\tau}_{i\alpha^2} u(k,\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk dk' \\
&= \frac{18\alpha\pi^{\frac{1}{2}}}{\nu^{\frac{1}{2}}} \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\grave{\tau}_{i\alpha^2} u(k,\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |u(k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&= \frac{18\alpha\pi}{\alpha\nu^{\frac{1}{2}}} \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |u(k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk' \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\grave{\tau}_{i\alpha^2} u(k,\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk \\
&= \frac{18\pi}{\alpha\nu^{\frac{1}{2}}} \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |u(k',\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk' \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |\grave{\tau}_{i\alpha^2} u(k,\omega')|^2 d\omega' \right]^{\frac{1}{2}} dk \\
&= \frac{18\pi^{\frac{1}{2}}}{\alpha\nu^{\frac{1}{2}}} ||u||_1 |||u|||_\alpha .
\end{aligned}
$$

We estimate also the function g and obtain

$$\begin{aligned}
|||g|||_\alpha &= \alpha \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\check{\tau}_{i\alpha^2} g(k,\omega)|^2 d\omega^{\frac{1}{2}} dk = \alpha \int_{\mathbb{R}^3} \int_{\mathbb{R}} |g(k,\omega - i\alpha^2)|^2 d\omega^{\frac{1}{2}} dk \\
&= \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left| \frac{f(k)}{i(\omega - i\alpha^2) + \nu |k|^2} \right|^2 d\omega \right]^{\frac{1}{2}} dk \\
&= \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left| \frac{f(k)}{i\omega - ii\alpha^2 + \nu |k|^2} \right|^2 d\omega \right]^{\frac{1}{2}} dk \\
&= \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left| \frac{f(k)}{i\omega + \alpha^2 + \nu |k|^2} \right|^2 d\omega \right]^{\frac{1}{2}} dk \\
&= \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \frac{|f(k)|^2}{|i\omega + \alpha^2 + \nu |k|^2|^2} d\omega \right]^{\frac{1}{2}} dk \\
&= \alpha \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \frac{|f(k)|^2}{\omega^2 + (\alpha^2 + \nu |k|^2)^2} d\omega \right]^{\frac{1}{2}} dk \\
&= \alpha \int_{\mathbb{R}^3} \left[\frac{|f(k)|^2}{\alpha^2 + \nu |k|^2} \int_{\mathbb{R}} \frac{1}{\omega'^2 + 1} d\omega' \right]^{\frac{1}{2}} dk \\
&= \alpha \int_{\mathbb{R}^3} \frac{\pi^{\frac{1}{2}} |f(k)|}{(\alpha^2 + \nu |k|^2)^{\frac{1}{2}}} dk \le \alpha \int_{\mathbb{R}^3} \frac{\pi^{\frac{1}{2}} |f(k)|}{\alpha} dk = \pi^{\frac{1}{2}} ||f||_1.
\end{aligned}$$

For indices α_1 and α_2, $\alpha_1 \le \alpha_2$ we have the inequality

$$\begin{aligned}
|||g|||_{\alpha_1} &= \alpha_1 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |g(k',\omega' - i\alpha_1^2)|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&= \alpha_1 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left| \frac{f(k')}{i(\omega' - i\alpha_1^2) + \nu |k'|^2} \right|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&= \alpha_1 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left| \frac{f(k')}{i\omega' - ii\alpha_1^2 + \nu |k'|^2} \right|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&= \alpha_1 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left| \frac{f(k')}{i\omega' + \alpha_1^2 + \nu |k'|^2} \right|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&= \alpha_1 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \frac{|f(k')|^2}{\omega'^2 + (\alpha_1^2 + \nu |k'|^2)^2} d\omega' \right]^{\frac{1}{2}} dk' \\
&\le \alpha_1 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \frac{|f(k')|^2}{\omega'^2 + (\alpha_1^2 + \frac{\alpha_1^2}{\alpha_2^2} \nu |k'|^2)^2} d\omega' \right]^{\frac{1}{2}} dk' \\
&= \alpha_1 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \frac{|f(k')|^2}{(\frac{\alpha_1^2}{\alpha_2^2}\omega'')^2 + (\alpha_1^2 + \frac{\alpha_1^2}{\alpha_2^2} \nu |k'|^2)^2} \frac{\alpha_1^2}{\alpha_2^2} d\omega'' \right]^{\frac{1}{2}} dk' \\
&= \alpha_1 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \frac{|f(k')|^2}{\omega''^2 + (\alpha_2^2 + \nu |k'|^2)^2} \frac{\alpha_2^2}{\alpha_1^2} d\omega'' \right]^{\frac{1}{2}} dk' \\
&= \alpha_2 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \frac{|f(k')|^2}{\omega'^2 + (\alpha_2^2 + \nu |k'|^2)^2} d\omega' \right]^{\frac{1}{2}} dk'
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_2 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left| \frac{f(k')}{i\omega' + \alpha_2^2 + \nu|k'|^2} \right|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&= \alpha_2 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left| \frac{f(k')}{i\omega' - ii\alpha_2^2 + \nu|k'|^2} \right|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&= \alpha_2 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} \left| \frac{f(k')}{i(\omega' - i\alpha_2^2) + \nu|k'|^2} \right|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&= \alpha_2 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}} |g(k', \omega' - i\alpha_2^2)|^2 d\omega' \right]^{\frac{1}{2}} dk' \\
&= |||g|||_{\alpha_1}.
\end{aligned}
$$

Although estimates (4.9) are nice, they do not verify the contractivity estimate of the Banach fixed point theorem. This is because the operator T has to be defined in a complete metric space D and such that $TD \subset D$, and this condition is still without proof.

4.3 Supremum-norm estimate

In this section we will write a supremum-norm estimate, that is based on the equation

$$
u(k,\omega) = -\hat{\Pi}^S(k) \sum_{n=1}^{3} \sum_{m=1}^{3} u_m * u_n(k,\omega) \frac{ie_n k_m}{i\omega + \nu|k|^2} + \frac{f(k)}{i\omega + \nu|k|^2}.
$$

We apply the change of variables $\tilde{u}(k,\omega) = (i\omega + \nu|k|^2)|k|^2 u(k,\omega)$ and obtain

$$
\begin{aligned}
\tilde{u}(k,\omega) = & -\hat{\Pi}^S(k) \sum_{n=1}^{3} \sum_{m=1}^{3} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \frac{\tilde{u}_m(k',\omega')}{|k'|^2(i\omega' + \nu|k'|^2)} \\
& \cdot \frac{\tilde{u}_n(k-k', \omega-\omega')}{|k-k'|^2(i(\omega-\omega') + \nu|k-k'|^2)} d\omega' dk' i e_n k_m |k|^2 + f(k)|k|^2.
\end{aligned}
$$

For notational convenience we write

$$
\begin{aligned}
u(k,\omega) = & -\hat{\Pi}^S(k) \sum_{n=1}^{3} \sum_{m=1}^{3} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \frac{u_m(k',\omega')}{|k'|^2(i\omega' + \nu|k'|^2)} \\
& \frac{u_n(k-k', \omega-\omega')}{|k-k'|^2(i(\omega-\omega') + \nu|k-k'|^2)} d\omega' dk' i e_n k_m |k|^2 + f(k)|k|^2
\end{aligned}
$$

without tilde. We define

$$
\begin{aligned}
Tu(k,\omega) = & -\hat{\Pi}^S(k) \sum_{n=1}^{3} \sum_{m=1}^{3} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \frac{u_m(k',\omega')}{|k'|^2(i\omega' + \nu|k'|^2)} \\
& \frac{u_n(k-k', \omega-\omega')}{|k-k'|^2(i(\omega-\omega') + \nu|k-k'|^2)} d\omega' dk' i e_n k_m\, |k|^2 \\
g(k,\omega) = & \; f(k)|k|^2.
\end{aligned}
$$

We notice, that the operator T is a singular integral operator. We obtain also

$$u = Tu + g. \tag{4.9}$$

Estimates will be based on the supremum-norm

$$||u|| = \sup_{k \in S, \omega \in \mathbb{R}} |u(k, \omega)|.$$

We now start to analyze properties of transformed heat kernel and establish an upper bound for absolute value estimated convolution. For that purpose we use the convolution formula

$$f * g = 2\pi \mathscr{F}^{-1}(\mathscr{F} f \mathscr{F} g). \tag{4.10}$$

We will apply Theorem 1.40 to establish the equation (4.10). Assume that $a_1 > 0$ and $a_2 > 0$. We define

$$\begin{aligned} f(\omega) &= \frac{1}{|\omega| + a_1} \\ g(\omega) &= \frac{1}{|\omega| + a_2}. \end{aligned}$$

Now there are bounded functions $f^1 \in L^1$ and $g^1 \in L^1$ such that

$$\begin{aligned} f(x_2) - f(x_1) &= \int_{x_1}^{x_2} f^1(t) dt \\ g(x_2) - g(x_1) &= \int_{x_1}^{x_2} g^1(t) dt. \end{aligned}$$

They are

$$\begin{aligned} f^1(\omega) &= -\mathrm{sgn}(\omega) \frac{1}{(|\omega| + a_1)^2} \\ g^1(\omega) &= -\mathrm{sgn}(\omega) \frac{1}{(|\omega| + a_2)^2}. \end{aligned}$$

We show this. The function f is continuous as a composite of continuous functions. Assume $x_1 > 0$ and $x_2 > 0$. We calculate

$$\begin{aligned} f(x_2) - f(x_1) &= \frac{1}{|x_2| + a_1} - \frac{1}{|x_1| + a_1} = \frac{1}{x_2 + a_1} - \frac{1}{x_1 + a_1} \\ &= \int_{x_1}^{x_2} \frac{d}{dx} \frac{1}{x + a_1} dx = \int_{x_1}^{x_2} -\frac{1}{(x + a_1)^2} dx \\ &= \int_{x_1}^{x_2} -\mathrm{sgn}(x) \frac{1}{(|x| + a_1)^2} dx = \int_{x_1}^{x_2} f^1(x) dx. \end{aligned}$$

Assume $x_1 < 0$ and $x_2 < 0$. We calculate

$$\begin{aligned} f(x_2) - f(x_1) &= \frac{1}{|x_2| + a_1} - \frac{1}{|x_1| + a_1} = \frac{1}{-x_2 + a_1} - \frac{1}{-x_1 + a_1} \\ &= \int_{x_1}^{x_2} \frac{d}{dx} \frac{1}{-x + a_1} dx = \int_{x_1}^{x_2} \frac{1}{(-x + a_1)^2} dx \\ &= \int_{x_1}^{x_2} -\mathrm{sgn}(x) \frac{1}{(|x| + a_1)^2} dx = \int_{x_1}^{x_2} f^1(x) dx. \end{aligned}$$

Assume $x_1 < 0$ and $x_2 > 0$. We calculate

$$\begin{aligned}
f(x_2) - f(x_1) &= f(x_2) - f(0) + f(0) - f(x_1) \\
&= \lim_{b \to 0^+} f(x_2) - \lim_{b \to 0^+} f(b) + \lim_{a \to 0^-} f(a) - \lim_{a \to 0^-} f(x_1) \\
&= \lim_{b \to 0^+} (f(x_2) - f(b)) + \lim_{a \to 0^-} (f(a) - f(x_1)) \\
&= \lim_{b \to 0^+} \int_b^{x_2} f^1(x)dx + \lim_{a \to 0^-} \int_{x_1}^{a} f^1(x)dx \\
&= \int_0^{x_2} f^1(x)dx + \int_{x_1}^{0} f^1(x)dx \\
&= \int_{x_1}^{0} f^1(x)dx + \int_0^{x_2} f^1(x)dx = \int_{x_1}^{x_2} f^1(x)dx.
\end{aligned}$$

Assume that $x_1 > 0$ and $x_2 < 0$. We calculate

$$f(x_2) - f(x_1) = -(f(x_1) - f(x_2)) = -\int_{x^2}^{x_1} f^1(x)dx = \int_{x_1}^{x_2} f^1(x)dx.$$

Hence for $x_1 \neq 0$ and $x_2 \neq 0$ we have

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f^1(x)dx.$$

For $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ we calculate

$$\begin{aligned}
f(x_2) - f(x_1) &= \lim_{a \to x_1} \lim_{b \to x_2} f(b) - \lim_{a \to x_1} \lim_{b \to x_2} f(a) = \lim_{a \to x_1} \lim_{b \to x_2} (f(b) - f(a)) \\
&= \lim_{a \to x_1} \lim_{b \to x_2} \int_a^b f^1(x)dx = \int_{x_1}^{x_2} f^1(x)dx.
\end{aligned}$$

Hence for every $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ we have

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f^1(x)dx.$$

We have

$$g(x_2) - g(x_1) = \int_{x_1}^{x_2} g^1(x)dx$$

respectively. Functions f^1 and g^1 are finitely piecewise continuous, that is, they are piecewise continuous with finite number of separate continuous pieces. In addition, they decrease quadratically. Hence they are in L^1.

Finitely piecewise continous L^1-functions can be approximated by compactly supported finitely piecewise continuous L^1-functions. Compactly supported finitely piecewise continuous L^1-functions can be approximated by compactly supported finitely piecewise constant functions in L^1 and compactly supported finitely piecewise constant functions by Lipschitz-continuous functions in L^1. Take a sequence g_n^1 of Lipschitz-continuous L^1-functions with Licpshitz-constants L_n such that $||g_n^1 - g^1||_1 \to 0$ as $n \to \infty$. We calculate

$$\begin{aligned}
|f^1 * g^1(b) - f^1 * g^1(a)| &= |f^1 * g^1(b) - f^1 * g_n^1(b) + f^1 * g_n^1(b) - f^1 * g_n^1(a) \\
&\quad + f^1 * g_n^1(a) - f^1 * g^1(a)| \\
&\leq |f^1 * g^1(b) - f^1 * g_n^1(b)| \\
&\quad + |f^1 * g_n^1(b) - f^1 * g_n^1(a)| \\
&\quad + |f^1 * g_n^1(a) - f^1 * g^1(a)| \\
&\leq \left| \int_{-\infty}^{\infty} f^1(t)(g^1(b-t) - g_n^1(b-t))dt \right| \\
&\quad + \left| \int_{-\infty}^{\infty} f^1(t)(g_n^1(b-t) - g_n^1(a-t))dt \right| \\
&\quad + \left| \int_{-\infty}^{\infty} f^1(t)(g_n^1(a-t) - g^1(a-t))dt \right| \\
&\leq ||f^1||_\infty ||g^1 - g_n^1||_1 + L_n|b-a| ||f^1||_1 \\
&\quad + ||f^1||_\infty ||g_n^1 - g^1||_1.
\end{aligned}$$

Fix $\epsilon > 0$. Fix n such that $||g^1 - g_n^1|| < \frac{\epsilon}{3||f^1||_\infty}$. Fix $a \in \mathbb{R}$. Fix $b \in \mathbb{R}$ such that $|b - a| < \frac{\epsilon}{3L_n||f^1||_1}$. Then we have

$$|f^1 * g^1(b) - f^1 * g^1(a)| < \epsilon.$$

This shows that $f^1 * g^1$ is continuous at a. Hence $f^1 * g^1$ is continuous. Convolutions $f^1 * g$ and $f * g$ exist by Schwartz inequality. We calculate applying these existence results, Fubini's theorem (Th 8.8 in [11]) and Theorem 1.9 to obtain

$$\begin{aligned}
f^1 * g(b) - f^1 * g(a) &= \int_{-\infty}^{\infty} f^1(t)g(b-t) - \int_{-\infty}^{\infty} f^1(t)g(a-t)dt \\
&= \int_{-\infty}^{\infty} (f^1(t)g(b-t) - f^1(t)g(a-t))dt \\
&= \int_{-\infty}^{\infty} f^1(t)(g(b-t) - g(a-t))dt \\
&= \int_{-\infty}^{\infty} f^1(t) \int_a^b g^1(x-t)dxdt \\
&= \int_{-\infty}^{\infty} \int_a^b f^1(t)g^1(x-t)dxdt \\
&= \int_a^b \int_{-\infty}^{\infty} f^1(t)g^1(x-t)dtdx \\
&= \int_a^b f^1 * g^1(x)dx, \\
(f^1 * g)'(t) &= \lim_{h \to 0} \frac{f^1 * g^1(t+h) - f^1 * g^1(t)}{h} \\
&= \frac{1}{h} \int_t^{t+h} f^1 * g^1(x)dx = f^1 * g^1(t)
\end{aligned}$$

$$(f^1 * g)' = f^1 * g^1, \tag{4.11}$$

$$\begin{aligned}
f * g(b) - f * g(a) &= g * f(b) - g * f(a) \\
&= \int_{-\infty}^{\infty} g(t)f(b-t) - \int_{-\infty}^{\infty} g(t)f(a-t)dt \\
&= \int_{-\infty}^{\infty} (g(t)f(b-t) - g(t)f(a-t))dt \\
&= \int_{-\infty}^{\infty} g(t)(f(b-t) - f(a-t))dt \\
&= \int_{-\infty}^{\infty} g(t) \int_a^b f^1(x-t)dxdt \\
&= \int_{-\infty}^{\infty} \int_a^b g(t)f^1(x-t)dxdt \\
&= \int_a^b \int_{-\infty}^{\infty} g(t)f^1(x-t)dtdx \\
&= \int_a^b g * f^1(x)dx = \int_a^b f^1 * g(x)dx,
\end{aligned}$$

$$\begin{aligned}
(f * g)'(t) &= \lim_{h \to 0} \frac{f^1 * g(t+h) - f^1 * g(t)}{h} \\
&= \frac{1}{h} \int_t^{t+h} f^1 * g(x)dx = f^1 * g(t)
\end{aligned}$$

$$(f * g)' = f^1 * g. \tag{4.12}$$

We will need equations (4.11) and (4.12) in the existence proof of $\mathscr{F}(f * g)$.

Lemma 4.5. *Assume that functions f and g are even and convolution $f * g$ exists. Then $f * g$ is even.*
Proof: We calculate

$$\begin{aligned}
f * g(-\omega) &= \int_{-\infty}^{\infty} f(\omega')g(-\omega - \omega')d\omega' = \int_{\infty}^{-\infty} f(-\omega')g(-\omega + \omega')(-d\omega') \\
&= \int_{-\infty}^{\infty} f(-\omega')g(-\omega + \omega')d\omega' = \int_{-\infty}^{\infty} f(-\omega')g(-(\omega - \omega'))d\omega' \\
&= \int_{-\infty}^{\infty} f(\omega')g(\omega - \omega')d\omega' = f * g(\omega)
\end{aligned}$$

and obtain the claim. □

Lemma 4.6. *Assume that even functions $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ have estimates*

$$\begin{aligned}
|f(\omega)| &\leq \frac{C}{1 + |\omega|}, \quad C > 0 \\
|g(\omega)| &\leq \frac{D}{1 + |\omega|}, \quad D > 0.
\end{aligned}$$

Then we have limits

$$\begin{aligned} f * g(\omega) \to 0 \quad as\ \omega &\to \infty \\ f * g(\omega) \to 0 \quad as\ \omega &\to -\infty. \end{aligned}$$

Proof: We calculate

$$\begin{aligned} \int_{-\infty}^{\infty} \left| \frac{1}{1+|\omega|} \right|^2 d\omega &= \int_{-\infty}^{\infty} \frac{1}{(1+|\omega|)^2} d\omega = \int_{-\infty}^{\infty} \frac{1}{1+2|\omega|+|\omega|^2} d\omega \\ &\leq \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega = \Big/_{-\infty}^{\infty} \arctan(\omega) \\ &= \frac{\pi}{2} - \left[-\frac{\pi}{2} \right] = \pi, \end{aligned}$$

$$\begin{aligned} \left[\int_{-\infty}^{\infty} |f(\omega')g(\omega-\omega')| d\omega' \right]^2 &\leq \int_{-\infty}^{\infty} |f(\omega')|^2 d\omega' \int_{-\infty}^{\infty} |g(\omega-\omega')|^2 d\omega' \\ &= \int_{-\infty}^{\infty} |f(\omega')|^2 d\omega' \int_{\infty}^{-\infty} |g(\omega')|^2 (-d\omega') \\ &= \int_{-\infty}^{\infty} |f(\omega')|^2 d\omega' \int_{-\infty}^{\infty} |g(\omega')|^2 d\omega' \\ &\leq \int_{-\infty}^{\infty} \left| \frac{C}{1+|\omega'|} \right|^2 d\omega' \int_{-\infty}^{\infty} \left| \frac{D}{1+|\omega'|} \right|^2 d\omega' \\ &= C^2 \int_{-\infty}^{\infty} \left| \frac{1}{1+|\omega'|} \right|^2 d\omega' D^2 \int_{-\infty}^{\infty} \left| \frac{1}{1+|\omega'|} \right|^2 d\omega' \\ &\leq C^2 D^2 \pi^2, \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} |f(\omega')g(\omega-\omega')| d\omega' &\leq CD\pi, \\ \int_{-\infty}^{\infty} f(\omega')g(\omega-\omega') d\omega' &\in \mathbb{C}. \end{aligned}$$

This shows that $f * g$ exists. Assume $\omega \geq 0$. We calculate

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{1+|\omega'|} \frac{1}{1+|\omega-\omega'|} d\omega' &= \int_{-\infty}^{-\frac{\omega}{2}} \frac{1}{1+|\omega'|} \frac{1}{1+|\omega-\omega'|} d\omega' \\ &+ \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \frac{1}{1+|\omega'|} \frac{1}{1+|\omega-\omega'|} d\omega' \\ &+ \int_{\frac{\omega}{2}}^{\frac{3\omega}{2}} \frac{1}{1+|\omega'|} \frac{1}{1+|\omega-\omega'|} d\omega' \\ &+ \int_{\frac{3\omega}{2}}^{\infty} \frac{1}{1+|\omega'|} \frac{1}{1+|\omega-\omega'|} d\omega' \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{-\frac{\omega}{2}} \frac{1}{1+|\omega'|}\frac{1}{1+|\omega-\omega'|}d\omega' + \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \frac{1}{1+|\omega'|}\frac{1}{1+|\omega-\omega'|}d\omega' \\
&\quad + \int_{\frac{\omega}{2}}^{-\frac{\omega}{2}} \frac{1}{1+|\omega-\omega'|}\frac{1}{1+|\omega'|}(-d\omega') + \int_{-\frac{\omega}{2}}^{-\infty} \frac{1}{1+|\omega-\omega'|}\frac{1}{1+|\omega'|}(-d\omega') \\
&= \int_{-\infty}^{-\frac{\omega}{2}} \frac{1}{1+|\omega'|}\frac{1}{1+|\omega-\omega'|}d\omega' + \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \frac{1}{1+|\omega'|}\frac{1}{1+|\omega-\omega'|}d\omega' \\
&\quad + \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \frac{1}{1+|\omega-\omega'|}\frac{1}{1+|\omega'|}d\omega' + \int_{-\infty}^{-\frac{\omega}{2}} \frac{1}{1+|\omega-\omega'|}\frac{1}{1+|\omega'|}d\omega' \\
&= 2\int_{-\infty}^{-\frac{\omega}{2}} \frac{1}{1+|\omega'|}\frac{1}{1+|\omega-\omega'|}d\omega' + 2\int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \frac{1}{1+|\omega'|}\frac{1}{1+|\omega-\omega'|}d\omega' \\
&\leq 2\int_{-\infty}^{-\frac{\omega}{2}} \frac{1}{1+|\omega'|}\frac{1}{1+|\omega'|}d\omega' + 2\int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \frac{1}{1+|\omega'|}\frac{1}{1+|\frac{\omega}{2}|}d\omega' \\
&= 2\left[\int_{-\infty}^{-\frac{\omega}{2}} \frac{1}{1+|\omega'|}\frac{1}{1+|\omega'|}d\omega' + \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \frac{1}{1+|\frac{\omega}{2}|}\frac{1}{1+|\omega'|}d\omega'\right] \\
&= 2\left[\int_{-\infty}^{-\frac{\omega}{2}} \frac{1}{1+2|\omega'|+|\omega'|^2}d\omega' + \frac{1}{1+|\frac{\omega}{2}|}\int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \frac{1}{1+|\omega'|}d\omega'\right] \\
&\leq 2\left[\int_{-\infty}^{-\frac{\omega}{2}} \frac{1}{1+\omega'^2}d\omega' + \frac{2}{1+|\frac{\omega}{2}|}\int_{0}^{\frac{\omega}{2}} \frac{1}{1+\omega'}d\omega'\right] \\
&= 2\left[\Big/_{-\infty}^{-\frac{\omega}{2}} \arctan(\omega') + \frac{2}{1+|\frac{\omega}{2}|}\Big/_{0}^{\frac{\omega}{2}} \ln(1+\omega')\right] \\
&= 2\left[\arctan(-\frac{\omega}{2}) - (-\frac{\pi}{2}) + \frac{2}{1+|\frac{\omega}{2}|}\ln(1+\frac{\omega}{2})\right] \\
&= 2\left[\frac{\pi}{2} - \arctan(\frac{\omega}{2}) + 2\ln(1+\frac{\omega}{2})e^{-\ln(1+\frac{\omega}{2})}\right] \to 0 \quad \text{as } \omega \to \infty.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
|f*g(\omega)| &= \left|\int_{-\infty}^{\infty} f(\omega')g(\omega-\omega')d\omega'\right| \leq \int_{-\infty}^{\infty} |f(\omega')g(\omega-\omega')|d\omega' \\
&= \int_{-\infty}^{\infty} |f(\omega')||g(\omega-\omega')|d\omega' \\
&\leq \int_{-\infty}^{\infty} \left|\frac{C}{1+|\omega'|}\right|\left|\frac{D}{1+|\omega-\omega'|}\right|d\omega' \\
&= CD\int_{-\infty}^{\infty} \frac{1}{1+|\omega'|}\frac{1}{1+|\omega-\omega'|}d\omega' \to 0 \quad \text{as } \omega \to \infty, \\
f*g(\omega) &\to 0 \quad \text{as } \omega \to \infty.
\end{aligned}
$$

Lemma 4.5 implies

$$
\begin{aligned}
f*g(-\omega) &= f*g(\omega) \to 0 \quad \text{as } \omega \to \infty, \\
f*g(\omega) &\to 0 \quad \text{as } \omega \to -\infty.
\end{aligned}
$$

This completes the proof. □

Functions f and g are even and they have estimates

$$\begin{aligned} |f(\omega)| &\leq \frac{C_1}{1+|\omega|}, \quad C_1 > 0, \\ |g(\omega)| &\leq \frac{C_2}{1+|\omega|}, \quad C_2 > 0. \end{aligned}$$

To show evenness of f and g, we calculate

$$f(\omega) = \frac{1}{|\omega| + a_1} = \frac{1}{|-\omega| + a_1} = f(-\omega).$$

The equation

$$g(\omega) = g(-\omega)$$

is obtained by replacing a_1 with a_2. To show the above estimate, we calculate

$$\begin{aligned} |f(\omega)| &= \frac{1}{|\omega| + a_1} = \frac{1+|\omega|}{1+|\omega|}\frac{1}{|\omega| + a_1} = \frac{1}{1+|\omega|}\frac{1+|\omega|}{|\omega| + a_1} \\ &\leq \frac{1}{1+|\omega|}\left[\frac{1}{|\omega| + a_1} + \frac{|\omega|}{|\omega| + a_1}\right] \\ &\leq \frac{1}{1+|\omega|}\left[\frac{1}{a_1} + \frac{|\omega| + a_1}{|\omega| + a_1}\right] \leq \frac{C_1}{1+|\omega|}. \end{aligned}$$

Now the estimate

$$|g(\omega)| \leq \frac{C_2}{1+|\omega|},$$

is obtained by replacing a_1 with a_2. Functions f^1 and g^1 have similar estimates. To show this, we calculate

$$\begin{aligned} |f^1(\omega)| &= \left| -\operatorname{sgn}(\omega)\frac{1}{(|\omega| + a_1)^2}\right| \leq \frac{1}{(|\omega| + a_1)^2} \\ &= \frac{1}{|\omega| + a_1}\frac{1}{|\omega| + a_1} \leq \frac{1}{a_1}|f(\omega)| \\ &\leq \frac{1}{a_1}\frac{C_1}{1+|\omega|} \leq \frac{D_1}{1+|\omega|}. \end{aligned}$$

Also the estimate

$$|g^1(\omega)| \leq \frac{D_2}{1+|\omega|}$$

is obtaind by replacing a_1 with a_2.

Lemma 4.7. *Assume that $f \in X$ and $g \in X$. Assume that there are functions $f^1 \in X$ and $g^1 \in X$ such that*

$$\begin{aligned} f(x_2) - f(x_1) &= \int_{x_1}^{x_2} f^1(t)dt, \\ g(x_2) - g(x_1) &= \int_{x_1}^{x_2} g^1(t)dt. \end{aligned}$$

Then we have the equation

$$\int_a^b f^1(x)g(x)dx = \Big/_a^b f(x)g(x) - \int_a^b f(x)g^1(x)dx.$$

Proof: We calculate

$$\begin{aligned}
\int_a^b f^1(x)g(x)dx &= \int_a^b f_1(x)(g(x)-g(a)+g(a))dx \\
&= \int_a^b (f^1(x)(g(x)-g(a))+f^1(x)g(a))dx \\
&= \int_a^b f^1(x)(g(x)-g(a))dx + \int_a^b f^1(x)g(a)dx \\
&= \int_a^b f^1(x)\int_a^x g^1(t)dtdx + \int_a^b f^1(x)g(a)dx \\
&= \int_a^b \int_a^x f^1(x)g^1(t)dtdx + \int_a^b f^1(x)g(a)dx \\
&= \int_a^b \int_t^b f^1(x)g^1(t)dxdt + \int_a^b f^1(x)g(a)dx \\
&= \int_a^b \int_t^b g^1(t)f^1(x)dxdt + \int_a^b f^1(x)g(a)dx \\
&= \int_a^b g^1(t)\int_t^b f^1(x)dxdt + \int_a^b f^1(x)g(a)dx \\
&= \int_a^b g^1(t)(f(b)-f(t))dt + \int_a^b f^1(x)g(a)dx \\
&= \int_a^b (g^1(t)f(b)-g^1(t)f(t))dt + \int_a^b f^1(x)g(a)dx \\
&= \int_a^b g^1(t)f(b)dt - \int_a^b g^1(t)f(t)dt + \int_a^b f^1(x)g(a)dx \\
&= \int_a^b g^1(t)f(b)dt + \int_a^b f^1(x)g(a)dx - \int_a^b g^1(t)f(t)dt \\
&= \int_a^b f(b)g^1(t)dt + \int_a^b g(a)f^1(x)dx - \int_a^b g^1(t)f(t)dt \\
&= f(b)\int_a^b g^1(t)dt + g(a)\int_a^b f^1(x)dx - \int_a^b g^1(t)f(t)dt \\
&= f(b)(g(b)-g(a)) + g(a)(f(b)-f(a)) - \int_a^b g^1(t)f(t)dt \\
&= f(b)g(b)-f(b)g(a)+g(a)f(b)-g(a)f(a) \\
&\quad -\int_a^b g^1(t)f(t)dt \\
&= f(b)g(b)-f(b)g(a)+f(b)g(a)-f(a)g(a) \\
&\quad -\int_a^b g^1(t)f(t)dt \\
&= f(b)g(b)-f(a)g(a)-\int_a^b g^1(t)f(t)dt
\end{aligned}$$

$$= \Big/_a^b f(x)g(x) - \int_a^b g^1(t)f(t)dt$$

to obtain the claim. □

Lemma 4.8. *Assume functions $f \in X$ and $g \in X$, $f^1 \in X$ and $g^1 \in X$ such that*

$$\begin{aligned} f(x_2) - f(x_1) &= \int_{x_1}^{x_2} f^1(t)dt, \\ g(x_2) - g(x_1) &= \int_{x_1}^{x_2} g^1(t)dt, \end{aligned}$$

functions f and g are bounded and they have limit 0 at ∞ and $-\infty$. Assume, that functions f^1 and g^1 are also absolutely integrable. Then we have the equation

$$\int_{-\infty}^{\infty} f^1(x)g(x)dx = \Big/_{-\infty}^{\infty} f(x)g(x) - \int_{-\infty}^{\infty} f(x)g^1(x)dx.$$

Proof: We estimate

$$\begin{aligned} \int_{-\infty}^{\infty} |f^1(x)g(x)|dx &= \int_{-\infty}^{\infty} |f^1(x)||g(x)|dx \leq \int_{-\infty}^{\infty} |f^1(x)|C_2dx \\ &= C_2\int_{-\infty}^{\infty} |f^1(x)|dx \leq C_2D_1, \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)g^1(x)|dx &= \int_{-\infty}^{\infty} |f(x)||g^1(x)|dx \leq \int_{-\infty}^{\infty} C_1|g^1(x)|dx \\ &= C_1\int_{-\infty}^{\infty} |g^1(x)|dx \leq C_1D_2. \end{aligned}$$

This shows that all integral terms in the equation converge absolutely and hence converge, that is they exists. The substitution term clearly exists. Hence we can calculate using Lemma 4.7 to obtain

$$\begin{aligned} \int_{-\infty}^{\infty} f^1(x)g(x)dx &= \lim_{\substack{M_1 \to -\infty \\ M_2 \to \infty}} \int_{M_1}^{M_2} f^1(x)g(x)dx \\ &= \lim_{\substack{M_1 \to -\infty \\ M_2 \to \infty}} \left[\Big/_{M_1}^{M_2} f(x)g(x) - \int_{M_1}^{M_2} f(x)g^1(x)dx\right] \\ &= \lim_{\substack{M_1 \to -\infty \\ M_2 \to \infty}} \Big/_{M_1}^{M_2} f(x)g(x) - \lim_{\substack{M_1 \to -\infty \\ M_2 \to \infty}} \int_{M_1}^{M_2} f(x)g^1(x)dx \\ &= \Big/_{-\infty}^{\infty} f(x)g(x) - \int_{-\infty}^{\infty} f(x)g^1(x)dx. \end{aligned}$$

This shows the claim. □

Assume $\omega \in [0, \infty)$. Fix $a > 0$. We find uniform convergence using Cauchy's criterion

$$
\begin{aligned}
\int_{M_1}^{M_2} e^{-(\omega+a)\tau} e^{i\omega t} d\tau &= \Big/_{M_1}^{M_2} -\frac{1}{\omega+a} e^{-(\omega+a)\tau} e^{i\omega t} \\
&= -\frac{1}{\omega+a}\left[e^{-(\omega+a)M_2} - e^{-(\omega+a)M_1}\right] e^{i\omega t} \\
&\to 0, \quad M_1, M_2 \to \infty
\end{aligned}
$$

for every $t \in \mathbb{R}$. Fix $t \neq 0$. We obtain

$$
\begin{aligned}
\int_M^{\infty} e^{-(\omega+a)\tau} e^{i\omega t} d\omega &= \int_M^{\infty} e^{-a\tau} e^{-\omega\tau + i\omega t} d\omega \\
&= e^{-a\tau} \Big/_M^{\infty} \frac{1}{-\tau + it} e^{-\omega\tau + i\omega t} \\
&= e^{-a\tau} \frac{1}{-\tau + it} (-e^{-M\tau + iMt}) \\
&= e^{-a\tau} \frac{1}{\tau - it} e^{-M\tau} e^{iMt}
\end{aligned}
$$

for every $\tau > 0$. Hence we have

$$
\begin{aligned}
\int_0^{M_1} \int_0^{\infty} e^{-(\omega+a)\tau} e^{i\omega t} d\tau d\omega &= \int_0^{M_1} \lim_{M_2 \to \infty} \int_0^{M_2} e^{-(\omega+a)\tau} e^{i\omega t} d\tau d\omega \\
&= \lim_{M_2 \to \infty} \int_0^{M_1} \int_0^{M_2} e^{-(\omega+a)\tau} e^{i\omega t} d\tau d\omega \\
&= \lim_{M_2 \to \infty} \int_0^{M_2} \int_0^{M_1} e^{-(\omega+a)\tau} e^{i\omega t} d\omega d\tau \\
&= \int_0^{\infty} \int_0^{M_1} e^{-(\omega+a)\tau} e^{i\omega t} d\omega d\tau \\
&= \lim_{a \to 0^+} \int_a^{\infty} \int_0^{M_1} e^{-(\omega+a)\tau} e^{i\omega t} d\omega d\tau \\
&= \lim_{a \to 0^+} \int_a^{\infty} \int_0^{\infty} e^{-(\omega+a)\tau} e^{i\omega t} d\omega d\tau \\
&\quad - \lim_{a \to 0^+} \int_a^{\infty} \int_{M_1}^{\infty} e^{-(\omega+a)\tau} e^{i\omega t} d\omega d\tau,
\end{aligned}
$$

$$
\begin{aligned}
&\left| \int_0^{M_1} \int_0^{\infty} e^{-(\omega+a)\tau} e^{i\omega t} d\tau d\omega - \lim_{a \to 0^+} \int_a^{\infty} \int_0^{\infty} e^{-(\omega+a)\tau} e^{i\omega t} d\omega d\tau \right| \\
&= \left| \lim_{a \to 0^+} \int_a^{\infty} \int_{M_1}^{\infty} e^{-(\omega+a)\tau} e^{i\omega t} d\omega d\tau \right| \\
&= \left| \lim_{a \to 0^+} \int_a^{\infty} e^{-a\tau} \frac{1}{\tau - it} e^{-M_1\tau} e^{iM_1 t} d\tau \right| \\
&= \left| \int_0^{\infty} e^{-a\tau} \frac{1}{\tau - it} e^{-M_1\tau} e^{iM_1 t} d\tau \right|
\end{aligned}
$$

$$\begin{aligned}
&\leq \int_0^\infty \left| e^{-a\tau}\frac{1}{\tau - it}e^{-M_1\tau}e^{iM_1t}\right| d\tau = \int_0^\infty \left|e^{-a\tau}\right|\left|\frac{1}{\tau-it}\right|\left|e^{-M_1\tau}\right|\left|e^{iM_1t}\right| d\tau \\
&= \int_0^\infty e^{-a\tau}\frac{1}{|\tau - it|}e^{-M_1\tau}d\tau \leq \int_0^\infty \frac{1}{|0-it|}e^{-M_1\tau}d\tau \\
&= \frac{1}{|0-it|}\int_0^\infty e^{-M_1\tau}d\tau = \frac{1}{|t|}\frac{1}{M_1} \to 0, \quad M_1 \to \infty.
\end{aligned}$$

According to the definition of convergence we have

$$\int_0^\infty\int_0^\infty e^{-(\omega+a)\tau}e^{i\omega t}d\tau d\omega = \int_0^\infty\int_0^\infty e^{-(\omega+a)\tau}e^{i\omega t}d\omega d\tau$$

for every $t \neq 0$. We calculate

$$\begin{aligned}
\int_0^\infty \frac{1}{\omega + a}e^{i\omega t}d\omega &= \int_0^\infty\int_0^\infty e^{-(\omega+a)\tau}d\tau e^{i\omega t}d\omega \\
&= \int_0^\infty\int_0^\infty e^{-(\omega+a)\tau}e^{i\omega t}d\tau d\omega \\
&= \int_0^\infty\int_0^\infty e^{-(\omega+a)\tau}e^{i\omega t}d\omega d\tau \\
&= \int_0^\infty\int_0^\infty e^{-a\tau}e^{-\omega\tau+i\omega t}d\omega d\tau \\
&= \int_0^\infty e^{-a\tau}\int_0^\infty e^{-\omega\tau+i\omega t}d\omega d\tau \\
&= \int_0^\infty e^{-a\tau}\Big/_0^\infty \frac{1}{-\tau+it}e^{-\omega\tau+i\omega t}d\tau \\
&= \int_0^\infty e^{-a\tau}\frac{1}{\tau - it}d\tau,\ a>0,\ t\neq 0,
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^0 \frac{1}{-\omega+a}e^{i\omega t}d\omega &= \int_\infty^0 \frac{1}{\omega+a}e^{i(-\omega)t}(-d\omega) = \int_0^\infty \frac{1}{\omega+a}e^{i(-\omega)t}d\omega \\
&= \int_0^\infty \frac{1}{\omega+a}e^{i\omega(-t)}d\omega = \int_0^\infty e^{-a\tau}\frac{1}{\tau - i(-t)}d\tau \\
&= \int_0^\infty e^{-a\tau}\frac{1}{\tau+it}d\tau,\ a>0,\ t\neq 0,
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^\infty \frac{1}{|\omega|+a}e^{i\omega t}d\omega &= \int_{-\infty}^0 \frac{1}{-\omega+a}e^{i\omega t}d\omega + \int_0^\infty \frac{1}{\omega+a}e^{i\omega t}d\omega \\
&= \int_0^\infty e^{-a\tau}\frac{1}{\tau+it}d\tau + \int_0^\infty e^{-a\tau}\frac{1}{\tau-it}d\tau \\
&= \int_0^\infty \left[e^{-a\tau}\frac{1}{\tau+it} + e^{-a\tau}\frac{1}{\tau-it}\right]d\tau \\
&= \int_0^\infty e^{-a\tau}\left[\frac{1}{\tau+it} + \frac{1}{\tau-it}\right]d\tau \\
&= \int_0^\infty e^{-a\tau}\frac{2\tau}{\tau^2+t^2}d\tau \\
&= 2\int_0^\infty e^{-a\tau}\frac{\tau}{\tau^2+t^2}d\tau
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^\infty e^{-a|t|\tau'}\frac{|t|\tau'}{(|t|\tau')^2+t^2}|t|d\tau' = 2\int_0^\infty e^{-a|t|\tau'}\frac{t^2\tau'}{t^2\tau'^2+t^2}d\tau' \\
&= 2\int_0^\infty e^{-a|t|\tau'}\frac{\tau'}{\tau'^2+1}d\tau',\ a>0,\ t\neq 0,
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{1}{|\omega|+a}e^{-i\omega t}d\omega &= \int_{-\infty}^{\infty}\frac{1}{|\omega|+a}e^{i\omega(-t)}d\omega \\
&= 2\int_0^\infty e^{-a|-t|\tau'}\frac{\tau'}{\tau'^2+1}d\tau' \\
&= 2\int_0^\infty e^{-a|t|\tau'}\frac{\tau'}{\tau'^2+1}d\tau',\ a>0,\ t\neq 0.
\end{aligned}
$$

This shows that $\mathscr{F}f$ exists. The existence of $\mathscr{F}g$ is shown in a similar way. Now we can establish the existence of $\mathscr{F}(f*g)$. We calculate

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f^1(\omega')g^1(\omega-\omega')e^{-it\omega}|d\omega d\omega' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f^1(\omega')||g^1(\omega-\omega')||e^{-it\omega}|d\omega d\omega' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f^1(\omega')||g^1(\omega-\omega')|d\omega d\omega' \\
&= \int_{-\infty}^{\infty}|f^1(\omega')|\int_{-\infty}^{\infty}|g^1(\omega-\omega')|d\omega d\omega' \\
&= \int_{-\infty}^{\infty}|f^1(\omega')|\int_{-\infty}^{\infty}|g^1(\omega)|d\omega d\omega' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g^1(\omega)|d\omega|f^1(\omega')|d\omega' \\
&= \int_{-\infty}^{\infty}|g^1(\omega)|d\omega\int_{-\infty}^{\infty}|f^1(\omega')|d\omega' \\
&= ||g^1||_1||f^1||_1 = ||f^1||_1||g^1||_1 \in \mathbb{R}.
\end{aligned}
$$

It is known that sets $\mathbb{R}$ and $\mathbb{R}$ are σ-finite measure spaces with Lebesgue measure. We apply Fubini's theorem (Th 8.8 in [11]) to obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty}f^1(\omega)e^{-it\omega}d\omega\int_{-\infty}^{\infty}g^1(\omega')e^{-it\omega'}d\omega' \\
&= \int_{-\infty}^{\infty}g^1(\omega)e^{-it\omega}d\omega\int_{-\infty}^{\infty}f^1(\omega')e^{-it\omega'}d\omega' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g^1(\omega)e^{-it\omega}d\omega f^1(\omega')e^{-it\omega'}d\omega' \\
&= \int_{-\infty}^{\infty}f^1(\omega')e^{-it\omega'}\int_{-\infty}^{\infty}g^1(\omega)e^{-it\omega}d\omega d\omega' \\
&= \int_{-\infty}^{\infty}f^1(\omega')e^{-it\omega'}\int_{-\infty}^{\infty}g^1(\omega-\omega')e^{-it(\omega-\omega')}d\omega d\omega'
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^1(\omega')e^{-it\omega'}g^1(\omega-\omega')e^{-it(\omega-\omega')}d\omega d\omega' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^1(\omega')e^{-it\omega'}g^1(\omega-\omega')e^{-it\omega+it\omega'}d\omega d\omega' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^1(\omega')g^1(\omega-\omega')e^{-it\omega+it\omega'}e^{-it\omega'}d\omega d\omega' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^1(\omega')g^1(\omega-\omega')e^{-it\omega+it\omega'-it\omega'}d\omega d\omega' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^1(\omega')g^1(\omega-\omega')e^{-it\omega}d\omega d\omega' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^1(\omega')g^1(\omega-\omega')e^{-it\omega}d\omega' d\omega \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it\omega}f^1(\omega')g^1(\omega-\omega')d\omega' d\omega \\
&= \int_{-\infty}^{\infty} e^{-it\omega}\int_{-\infty}^{\infty} f^1(\omega')g^1(\omega-\omega')d\omega' d\omega \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^1(\omega')g^1(\omega-\omega')d\omega' e^{-it\omega}d\omega \\
&= \int_{-\infty}^{\infty} f^1 * g^1(\omega)e^{-it\omega}d\omega \\
&= \int_{-\infty}^{\infty} (f^1 * g)'(\omega)e^{-it\omega}d\omega \\
&= \Big/_{-\infty}^{\infty} f^1 * g(\omega)e^{-it\omega} - \int_{-\infty}^{\infty} f^1 * g(\omega)(-it)e^{-it\omega}d\omega \\
&= -\int_{-\infty}^{\infty} (f * g)'(\omega)(-it)e^{-it\omega}d\omega \\
&= -\left[\Big/_{-\infty}^{\infty} f * g(\omega)(-it)e^{-it\omega} - \int_{-\infty}^{\infty} f * g(\omega)(-it)(-it)e^{-it\omega}d\omega\right] \\
&= -\Big/_{-\infty}^{\infty} f * g(\omega)(-it)e^{-it\omega} + \int_{-\infty}^{\infty} f * g(\omega)(-it)(-it)e^{-it\omega}d\omega \\
&= \int_{-\infty}^{\infty} f * g(\omega)(-it)^2e^{-it\omega}d\omega \\
&= (-it)^2\int_{-\infty}^{\infty} f * g(\omega)e^{-it\omega}d\omega,\ t \neq 0.
\end{aligned}
$$

This shows that $\mathscr{F}(f * g)$ exists. We obtain also

$$
\begin{aligned}
(-it)^2\int_{-\infty}^{\infty} f * g(\omega)e^{-i\omega t}d\omega &= \int_{-\infty}^{\infty} f^1(\omega')e^{-it\omega'}d\omega'\int_{-\infty}^{\infty} g^1(\omega)e^{-it\omega}d\omega \\
&= \left[-\int_{-\infty}^{\infty} f^1(\omega')e^{-it\omega'}d\omega'\right]\left[-\int_{-\infty}^{\infty} g^1(\omega)e^{-it\omega}d\omega\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\diagup\!\!\!\int_{-\infty}^{\infty} f(\omega')e^{-it\omega'} - \int_{-\infty}^{\infty} f^1(\omega')e^{-it\omega'}d\omega'\right] \\
&\quad \left[\diagup\!\!\!\int_{-\infty}^{\infty} g(\omega)e^{-it\omega} - \int_{-\infty}^{\infty} g^1(\omega)e^{-it\omega}d\omega\right] \\
&= \int_{-\infty}^{\infty} f(\omega)(-it)e^{-it\omega}d\omega \int_{-\infty}^{\infty} g(\omega')(-it)e^{-it\omega'}d\omega' \\
&= (-it)\int_{-\infty}^{\infty} f(\omega)e^{-it\omega}d\omega(-it)\int_{-\infty}^{\infty} g(\omega')e^{-it\omega'}d\omega' \\
&= (-it)(-it)\int_{-\infty}^{\infty} f(\omega)e^{-it\omega}d\omega\int_{-\infty}^{\infty} g(\omega')e^{-it\omega'}d\omega' \\
&= (-it)^2\int_{-\infty}^{\infty} f(\omega)e^{-it\omega}d\omega\int_{-\infty}^{\infty} g(\omega')e^{-it\omega'}d\omega', \quad t \neq 0,
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} f*g(\omega)e^{-i\omega t}d\omega &= \int_{-\infty}^{\infty} f(\omega)e^{-it\omega}d\omega\int_{-\infty}^{\infty} g(\omega')e^{-it\omega'}d\omega', \quad t \neq 0 \\
\mathscr{F}(f*g)(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f*g(\omega)e^{-i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{-it\omega}d\omega\int_{-\infty}^{\infty} g(\omega')e^{-it\omega'}d\omega' \\
&= \int_{-\infty}^{\infty} f(\omega)e^{-it\omega}d\omega\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega')e^{-it\omega'}d\omega' \\
&= 2\pi\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{-it\omega}d\omega\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega')e^{-it\omega'}d\omega' \\
&= 2\pi\mathscr{F}f(t)\mathscr{F}g(t) = 2\pi\mathscr{F}f\mathscr{F}g(t), \quad t \neq 0.
\end{aligned}
$$

To establish the integrability of $|\mathscr{F}(f*g)|$ we estimate

$$
\begin{aligned}
&\int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\left|e^{-(\tau'a_1+\tau''a_2)|t|}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}\right|dtd\tau'd\tau'' \\
&= \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} e^{-(\tau'a_1+\tau''a_2)|t|}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}dtd\tau'd\tau'' \\
&= \int_0^{\infty}\int_0^{\infty}\frac{2}{\tau'a_1+\tau''a_2}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau'd\tau'' \\
&= 2\int_0^{\infty}\int_0^{\infty}\frac{1}{\sqrt{\tau'a_1+\tau''a_2}^2}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau'd\tau'' \\
&\le 2\int_0^{\infty}\int_0^{\infty}\frac{1}{\sqrt{\tau'a_1}\sqrt{\tau''a_2}}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau'd\tau'' \\
&\le \frac{2}{\sqrt{a_1a_2}}\int_0^{\infty}\int_0^{\infty}\frac{\tau'^{\frac{1}{2}}}{\tau'^2+1}\frac{\tau''^{\frac{1}{2}}}{\tau''^2+1}d\tau'd\tau'' \\
&= \frac{2}{\sqrt{a_1a_2}}\int_0^{\infty}\frac{\tau^{\frac{1}{2}}}{\tau^2+1}d\tau^2 \in \mathbb{R}.
\end{aligned}
$$

The variable t defines real line, that is known to be σ-finite measure space with Lebesgue measure. Variables τ' and τ'' define a plane $[0,\infty)\times[0,\infty)$, that is

σ-compact set, because it is a countable union of compact squares. The explicit construction is similar to that of $\mathbb{N}^2$. Hence the plane is also a σ-finite measure space with Lebesgue measure. The integrand is continuous on $\mathbb{R} \times [0,\infty)^2$. Hence it is also measurable on $\mathbb{R} \times [0,\infty)^2$. We calculate using Fubini's theorem (Theorem 8.8 in [11]) to obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f * g(\omega) e^{-i\omega t} d\omega \right| dt \\
&= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(\omega') e^{-i\omega' t} d\omega' \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega \right| dt \\
&= \int_{-\infty}^{\infty} \left| 2\int_{0}^{\infty} e^{-a_1|t|\tau'} \frac{\tau'}{\tau'^2+1} d\tau' \, 2\int_{0}^{\infty} e^{-a_2|t|\tau''} \frac{\tau''}{\tau''^2+1} d\tau'' \right| dt \\
&= 4\int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{-(\tau' a_1 + \tau'' a_2)|t|} \frac{\tau'}{\tau'^2+1} \frac{\tau''}{\tau''^2+1} d\tau' d\tau'' dt \\
&= 4\int_{0}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-(\tau' a_1 + \tau'' a_2)|t|} \frac{\tau'}{\tau'^2+1} \frac{\tau''}{\tau''^2+1} dt d\tau' d\tau'' \\
&= 4\int_{0}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} \left| e^{-(\tau' a_1 + \tau'' a_2)|t|} \frac{\tau'}{\tau'^2+1} \frac{\tau''}{\tau''^2+1} \right| dt d\tau' d\tau'' \\
&\leq 4\frac{2}{\sqrt{a_1 a_2}} \int_{0}^{\infty} \frac{\tau^{\frac{1}{2}}}{\tau^2+1} d\tau^2 \in \mathbb{R}, \quad a_1 > 0, \quad a_2 > 0.
\end{aligned}$$

This shows that $|\mathscr{F}(f * g)|$ is Riemann-integrable. Hence $x \in A$. Also the assumption 1 of Theorem 1.40 is verified for $x = f * g$. Lemma 4.6 shows, that x satisfies the assumption 2. For positive values of ω we calculate

$$\begin{aligned}
(f * g)'(\omega) &= f * g^1(\omega) = g^1 * f(\omega) = \int_{-\infty}^{\infty} g^1(\omega') f(\omega - \omega') d\omega' \\
&= \int_{-\infty}^{\infty} f(\omega - \omega') g^1(\omega') d\omega' \\
&= \int_{-\infty}^{\infty} \frac{1}{|\omega - \omega'| + a_1} \frac{-\mathrm{sgn}(\omega')}{(|\omega'| + a_2)^2} d\omega' \\
&= \int_{-\infty}^{\infty} \frac{-1}{|\omega - \omega'| + a_1} \frac{\mathrm{sgn}(\omega')}{(|\omega'| + a_2)^2} d\omega' \\
&= \int_{-\infty}^{0} \frac{-1}{|\omega - \omega'| + a_1} \frac{\mathrm{sgn}(\omega')}{(|\omega'| + a_2)^2} d\omega' \\
&\quad + \int_{0}^{\infty} \frac{-1}{|\omega - \omega'| + a_1} \frac{\mathrm{sgn}(\omega')}{(|\omega'| + a_2)^2} d\omega' \\
&\leq \int_{\infty}^{0} \frac{-1}{|\omega + \omega'| + a_1} \frac{\mathrm{sgn}(-\omega')}{(|-\omega'| + a_2)^2} (-d\omega') \\
&\quad + \int_{0}^{\infty} \frac{-1}{\omega + \omega' + a_1} \frac{\mathrm{sgn}(\omega')}{(|\omega'| + a_2)^2} d\omega' \\
&= \int_{0}^{\infty} \frac{1}{|\omega + \omega'| + a_1} \frac{\mathrm{sgn}(\omega')}{(|\omega'| + a_2)^2} d\omega' \\
&\quad + \int_{0}^{\infty} \frac{-1}{|\omega + \omega'| + a_1} \frac{\mathrm{sgn}(\omega')}{(|\omega'| + a_2)^2} d\omega' \\
&= 0.
\end{aligned}$$

For negative values of ω we calculate

$$\begin{aligned} f * g(\omega) &= f * g(-\omega), \quad \omega < 0 \\ (f * g)'(\omega) &= -(f * g)'(-\omega) \geq 0, \quad \omega < 0. \end{aligned}$$

To check the assumption 3 we calculate

$$\begin{aligned} \int_{-\infty}^{\infty} |(f * g)'(\omega)| d\omega &= \int_{-\infty}^{0} |(f * g)'(\omega)| d\omega + \int_{0}^{\infty} |(f * g)'(\omega)| d\omega \\ &= \int_{-\infty}^{0} (f * g)'(\omega) d\omega - \int_{0}^{\infty} (f * g)'(\omega) d\omega \\ &= \Big/_{-\infty}^{0} f * g(\omega) - \Big/_{0}^{\infty} f * g(\omega) \\ &= (f * g(0) - 0) - (0 - f * g(0)) \\ &= 2\,(f * g)(0). \end{aligned}$$

Theorem 1.40 now implies

$$\begin{aligned} (f * g)(\omega) &= \int_{-\infty}^{\infty} \mathscr{F}(f * g)(t) e^{i\omega t} dt = \int_{-\infty}^{\infty} 2\pi \mathscr{F} f \mathscr{F} g(t) e^{i\omega t} dt \\ &= 2\pi \int_{-\infty}^{\infty} \mathscr{F} f \mathscr{F} g(t) e^{i\omega t} dt = 2\pi \mathscr{F}^{-1}(\mathscr{F} f \mathscr{F} g)(\omega) \\ f * g &= 2\pi \mathscr{F}^{-1}(\mathscr{F} f \mathscr{F} g). \end{aligned}$$

This is the equation (4.10). It is worth noting that transforms $\mathscr{F} f$ and $\mathscr{F} g$ are singular at 0. Hence they are always integrated using improper Riemann-integral of the kind considered in [6]. Assume $a \neq 0$ or $b \neq 0$. We estimate

$$\begin{aligned} \frac{1}{|a+ib|} &= \frac{|a|+|b|}{|a+ib|} \frac{1}{|a|+|b|} = \left[\frac{|a|}{|a+ib|} + \frac{|b|}{|a+ib|} \right] \frac{1}{|a|+|b|} \\ &\leq (1+1) \frac{1}{|a|+|b|} = \frac{2}{|a|+|b|} \\ \frac{1}{|ib+a|} &= \frac{1}{|a+ib|} \leq \frac{2}{|a|+|b|} = \frac{2}{|b|+|a|}. \end{aligned}$$

To establish a change of order of integration, we estimate

$$\begin{aligned} &\int_{0}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} \left| e^{-(\tau' a_1 + \tau'' a_2)|t|} e^{-i\omega t} \frac{\tau'}{\tau'^2+1} \frac{\tau''}{\tau''^2+1} \right| dt d\tau' d\tau'' \\ &= \int_{0}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} \left| e^{-(\tau' a_1 + \tau'' a_2)|t|} \right| \left| e^{i\omega t} \right| \left| \frac{\tau'}{\tau'^2+1} \right| \left| \frac{\tau''}{\tau''^2+1} \right| dt d\tau' d\tau'' \\ &= \int_{0}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-(\tau' a_1 + \tau'' a_2)|t|} \frac{\tau'}{\tau'^2+1} \frac{\tau''}{\tau''^2+1} dt d\tau' d\tau'' \\ &\leq \frac{2}{\sqrt{a_1 a_2}} \int_{0}^{\infty} \frac{\tau^{\frac{1}{2}}}{\tau^2+1} d\tau^2 \in \mathbb{R}. \end{aligned}$$

The variable t defines real line, that is known to be σ-finite measure space with Lebesgue measure. Variables τ' and τ'' define a plane $[0,\infty)\times[0,\infty)$, that is also a σ-finite measure space with Lebesgue measure. The integrand is continuous on $\mathbb{R}\times[0,\infty)^2$. Hence it is also measurable on $\mathbb{R}\times[0,\infty)^2$. We calculate applying Fubini's theorem (Theorem 8.8 in [11]), Example 11.5 in [8] and Table 11.1 in [8] to obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{|i\omega'+a_1|}\frac{1}{|i(\omega-\omega')+a_2|}d\omega'\\
\leq\;& \int_{-\infty}^{\infty}\frac{2}{|\omega'|+a_1}\frac{2}{|\omega-\omega'|+a_2}d\omega'\\
=\;& 4(f*g)(\omega)=4\cdot 2\pi\mathscr{F}^{-1}(\mathscr{F}f\mathscr{F}g)(\omega)\\
=\;& 4\cdot 2\pi\int_{-\infty}^{\infty}(\mathscr{F}f\mathscr{F}g)(t)e^{i\omega t}dt=8\pi\int_{-\infty}^{\infty}\mathscr{F}f(t)\mathscr{F}g(t)e^{i\omega t}dt\\
=\;& 8\pi\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\omega'')e^{-it\omega''}d\omega''\frac{1}{2\pi}\int_{-\infty}^{\infty}g(\omega')e^{-it\omega'}d\omega'e^{i\omega t}dt\\
=\;& \frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\omega'')e^{-it\omega''}d\omega''\int_{-\infty}^{\infty}g(\omega')e^{-it\omega'}d\omega'e^{i\omega t}dt\\
=\;& \frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{|\omega''|+a_1}e^{-it\omega''}d\omega''\int_{-\infty}^{\infty}\frac{1}{|\omega'|+a_2}e^{-it\omega'}d\omega'e^{i\omega t}dt\\
=\;& \frac{2}{\pi}\int_{-\infty}^{\infty}2\int_{0}^{\infty}e^{-a_1|t|\tau'}\frac{\tau'}{\tau'^2+1}d\tau'\,2\int_{0}^{\infty}e^{-a_2|t|\tau''}\frac{\tau''}{\tau''^2+1}d\tau''e^{i\omega t}dt\\
=\;& \frac{8}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}e^{-a_1|t|\tau'-a_2|t|\tau''}e^{i\omega t}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau'd\tau''dt\\
=\;& \frac{8}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}e^{-(\tau'a_1+\tau''a_2)|t|}e^{i\omega t}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau'd\tau''dt\\
=\;& \frac{8}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}e^{-(\tau'a_1+\tau''a_2)|t|}e^{i\omega t}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}dtd\tau'd\tau''\\
=\;& \frac{8}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}e^{-(\tau'a_1+\tau''a_2)|t|}e^{i\omega t}dt\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau'd\tau''\\
=\;& 16\int_{0}^{\infty}\int_{0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-(\tau'a_1+\tau''a_2)|t|}e^{-i\omega t}dt\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau'd\tau''\\
=\;& 16\int_{0}^{\infty}\int_{0}^{\infty}\frac{1}{\pi}\frac{1}{|\tau'a_1+\tau''a_2|}\frac{1}{1+(\frac{\omega}{\tau'a_1+\tau''a_2})^2}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau'd\tau''\\
=\;& \frac{16}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}\frac{\tau'a_1+\tau''a_2}{(\tau'a_1+\tau''a_2)^2+\omega^2}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau'd\tau''\\
\leq\;& \frac{16}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}\frac{1}{\tau'a_1+\tau''a_2}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau'd\tau''.
\end{aligned}
$$

To establish another integrablity result we calculate

$$
I \;=\; \int_0^{\frac{\pi}{2}}\frac{1}{a^2\cos(\theta)+b^2\sin(\theta)}d\theta=\int_0^1\frac{1}{a^2\frac{1-t^2}{1+t^2}+b^2\frac{2t}{1+t^2}}\frac{2dt}{1+t^2}
$$

$$
\begin{aligned}
&= \int_0^1 \frac{1}{a^2(1-t^2)+2b^2t}2dt = \frac{2}{a^2}\int_0^1 \frac{1}{1-t^2+\frac{2b^2}{a^2}t}dt \\
&= \frac{2}{a^2}\int_0^1 \frac{1}{-t^2+\frac{2b^2}{a^2}t-\frac{b^4}{a^4}+\frac{b^4}{a^4}+1}dt \\
&= \frac{2}{a^2}\int_0^1 \frac{1}{\frac{b^4}{a^4}+1-(t-\frac{b^2}{a^2})^2}dt = \frac{2}{a^2}\frac{1}{\frac{b^4}{a^4}+1}\int_0^1 \frac{1}{1-(\frac{t-\frac{b^2}{a^2}}{\sqrt{\frac{b^4}{a^4}+1}})^2}dt \\
&= \frac{2}{a^2}\frac{1}{\sqrt{\frac{b^4}{a^4}+1}}\Big/_0^1 \operatorname{artanh}(\frac{t-\frac{b^2}{a^2}}{\sqrt{\frac{b^4}{a^4}+1}}) \\
&= \frac{2}{a^2}\frac{1}{\sqrt{\frac{b^4}{a^4}+\frac{a^4}{a^4}}}(\operatorname{artanh}(\frac{\frac{a^2}{a^2}-\frac{b^2}{a^2}}{\sqrt{\frac{b^4}{a^4}+\frac{a^4}{a^4}}})-\operatorname{artanh}(\frac{-\frac{b^2}{a^2}}{\sqrt{\frac{b^4}{a^4}+\frac{a^4}{a^4}}})) \\
&= \frac{2(\operatorname{artanh}(\frac{a^2-b^2}{\sqrt{a^4+b^4}})+\operatorname{artanh}(\frac{b^2}{\sqrt{a^4+b^4}}))}{\sqrt{a^4+b^4}}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
J &= \int_0^{\frac{\pi}{2}}(\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)}+\frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)})d\theta \\
&= \frac{2(\operatorname{artanh}(\frac{|k'|^2-|k-k'|^2}{\sqrt{|k'|^4+|k-k'|^4}})+\operatorname{artanh}(\frac{|k-k'|^2}{\sqrt{|k'|^4+|k-k'|^4}}))}{\sqrt{|k'|^4+|k-k'|^4}} \\
&\quad+\frac{2(\operatorname{artanh}(\frac{|k-k'|^2-|k'|^2}{\sqrt{|k-k'|^4+|k'|^4}})+\operatorname{artanh}(\frac{|k'|^2}{\sqrt{|k-k'|^4+|k'|^4}}))}{\sqrt{|k-k'|^4+|k'|^2}} \\
&= \frac{2(\operatorname{artanh}(\frac{|k'|^2}{\sqrt{|k'|^4+|k-k'|^4}})+\operatorname{artanh}(\frac{|k-k'|^2}{\sqrt{|k'|^4+|k-k'|^4}}))}{\sqrt{|k'|^4+|k-k'|^4}}.
\end{aligned}
$$

We estimate further

$$
\begin{aligned}
|k'|^4+|k-k'|^4 &= |k'-\frac{1}{2}k+\frac{1}{2}k|^4+|k'-\frac{1}{2}k-\frac{1}{2}k|^4 \\
&= (|k'-\frac{1}{2}k|^2+2(k'-\frac{1}{2}k)\cdot\frac{1}{2}k+|\frac{1}{2}k|^2)^2 \\
&\quad+(|k'-\frac{1}{2}k|^2-2(k'-\frac{1}{2}k)\cdot\frac{1}{2}k+|\frac{1}{2}k|^2)^2 \\
&= (|k'-\frac{1}{2}k|^2+|\frac{1}{2}k|^2)^2 \\
&\quad+2(|k'-\frac{1}{2}k|^2+|\frac{1}{2}k|^2)\cdot 2(k'-\frac{1}{2}k)\cdot\frac{1}{2}k \\
&\quad+(2(k'-\frac{1}{2}k)\cdot\frac{1}{2}k)^2 \\
&\quad+(|k'-\frac{1}{2}k|^2+|\frac{1}{2}k|^2)^2 \\
&\quad-2(|k'-\frac{1}{2}k|^2+|\frac{1}{2}k|^2)\cdot 2(k'-\frac{1}{2}k)\cdot\frac{1}{2}k
\end{aligned}
$$

$$
\begin{aligned}
&+(2(k'-\frac{1}{2}k)\cdot\frac{1}{2}k)^2 \\
=\;& 2(|k'-\frac{1}{2}k|^2+|\frac{1}{2}k|^2)^2+2(2(k'-\frac{1}{2}k)\cdot\frac{1}{2}k)^2 \\
\geq\;& 2|\frac{1}{2}k|^4 \geq \frac{1}{16}|k|^4
\end{aligned}
$$

The first area tangens hyperbolica term in the integral J has singularity at $k=k'$. To ensure, that it is integrable, we estimate

$$
\begin{aligned}
\operatorname{artanh}(\frac{a^2}{\sqrt{a^4+b^4}}) &= \frac{1}{2}\ln(\frac{1+\frac{a^2}{\sqrt{a^4+b^4}}}{1-\frac{a^2}{\sqrt{a^4+b^4}}}) = \frac{1}{2}\ln(\frac{\sqrt{a^4+b^4}+a^2}{\sqrt{a^4+b^4}-a^2}) \\
&= \frac{1}{2}\ln(\frac{(\sqrt{a^4+b^4}+a^2)(\sqrt{a^4+b^4}+a^2)}{(\sqrt{a^4+b^4}-a^2)(\sqrt{a^4+b^4}+a^2)}) \\
&= \frac{1}{2}\ln(\frac{(\sqrt{a^4+b^4}+a^2)^2}{a^4+b^4-a^4}) \\
&= \ln(\frac{\sqrt{a^4+b^4}+a^2}{b^2}) \\
&\leq \ln(\frac{\sqrt{a^4+2a^2b^2+b^4}+a^2+b^2}{b^2}) \\
&= \ln(2\frac{a^2+b^2}{b^2}) \leq \ln(2\frac{a^2+2ab+b^2}{b^2}) \\
&\leq \ln(4\frac{(a+b)^2}{b^2}) = 2\ln(2\frac{a+b}{b}) \\
&= 2\ln(2)+2\ln(\frac{a+b}{b}) \\
&\leq 2+2\ln(\frac{a+2\sqrt{a}\sqrt{b}+b}{b}) \\
&= 2+2\ln(\frac{(\sqrt{a}+\sqrt{b})^2}{b}) \\
&= 2+4\ln(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{b}}) \\
&= 2+4\ln(1+\frac{\sqrt{a}}{\sqrt{b}}) \leq 2+4\frac{\sqrt{a}}{\sqrt{b}}.
\end{aligned}
$$

For the first area tangens hyperbolica term we obtain

$$
\operatorname{artanh}(\frac{|k'|^2}{\sqrt{|k'|^4+|k-k'|^4}}) \leq 2+4\frac{\sqrt{|k'|}}{\sqrt{|k-k'|}}.
$$

The second term has singularity at $k'=0$. We apply commutativity of sum to obtain

$$
\operatorname{artanh}(\frac{|k-k'|^2}{\sqrt{|k'|^4+|k-k'|^4}}) = \operatorname{artanh}(\frac{|k-k'|^2}{\sqrt{|k-k'|^4+|k'|^2}}) \leq 2+4\frac{\sqrt{|k-k'|}}{\sqrt{|k'|}}.
$$

Assume $0<\alpha<3$ and $0<\beta<3$ such that $\alpha+\beta=4$. For $|k'|<\frac{1}{2}|k|$ we estimate

$$
\begin{aligned}
\int_{|k'|<\frac{1}{2}|k|} \frac{1}{|k'|^{\alpha}} \frac{1}{|k-k'|^{\beta}} \frac{1}{|k|^2} dk' &\leq \int_{|k'|<\frac{1}{2}|k|} \frac{1}{|k'|^{\alpha}} \frac{1}{||k|-|k'||^{\beta}} \frac{1}{|k|^2} dk' \\
&= \int_{|k'|<\frac{1}{2}|k|} \frac{1}{|k'|^{\alpha}} \frac{1}{(|k|-|k'|)^{\beta}} \frac{1}{|k|^2} dk'
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{|k'|<\frac{1}{2}|k|} \frac{1}{|k'|^{\alpha}} \frac{1}{(|k|-\frac{1}{2}|k|)^{\beta}} \frac{1}{|k|^2} dk' \leq \int_{|k'|<\frac{1}{2}|k|} \frac{1}{|k'|^{\alpha}} \frac{1}{(\frac{1}{2}|k|)^{\beta}} \frac{1}{|k|^2} dk' \\
&= \frac{2^{\beta}}{|k|^{\beta+2}} \int_{|k'|<\frac{1}{2}|k|} \frac{1}{|k'|^{\alpha}} dk' = \frac{2^{\beta}}{|k|^{\beta+2}} \int_0^{\frac{1}{2}|k|} \frac{1}{r^{\alpha}} 4\pi r^2 dr \\
&= \frac{4\pi \cdot 2^{\beta}}{|k|^{\beta+2}} \int_0^{\frac{1}{2}|k|} r^{2-\alpha} dr = \frac{4\pi \cdot 2^{\beta}}{|k|^{\beta+2}} \Big/_0^{\frac{1}{2}|k|} \frac{r^{3-\alpha}}{3-\alpha} = \frac{4\pi \cdot 2^{\beta}}{|k|^{\beta+2}} \frac{(\frac{1}{2}|k|)^{3-\alpha}}{3-\alpha} \\
&= \frac{4\pi \cdot 2^{\beta}}{|k|^{\beta+2}} \frac{1}{3-\alpha} \frac{2^{\alpha-3}}{|k|^{\alpha-3}} = \frac{4\pi}{3-\alpha} \frac{2^{\alpha+\beta-3}}{|k|^{\alpha+\beta-1}} = \frac{4\pi}{3-\alpha} \frac{2^{4-3}}{|k|^{4-1}} \\
&= \frac{8\pi}{(3-\alpha)} \frac{1}{|k|^3} \in \mathbb{R} \quad \forall k \neq 0.
\end{aligned}
$$

For $k' \in \mathbb{R}^3$, $|k'-k| \leq \frac{1}{2}|k|$ we estimate

$$
\begin{aligned}
\int_{|k'-k|<\frac{1}{2}|k|} \frac{1}{|k'|^{\alpha}} \frac{1}{|k-k'|^{\beta}} \frac{1}{|k|^2} dk' &\leq \int_{|-k'|<\frac{1}{2}|k|} \frac{1}{|k-k'|^{\alpha}} \frac{1}{|k'|^{\beta}} \frac{1}{|k|^2} dk' \\
&\leq \int_{|k'|<\frac{1}{2}|k|} \frac{1}{|k'|^{\beta}} \frac{1}{|k-k'|^{\alpha}} \frac{1}{|k|^2} dk' \\
&\leq \frac{8\pi}{(3-\beta)} \frac{1}{|k|^3} \in \mathbb{R} \quad \forall k \neq 0
\end{aligned}
$$

For $|k'| > \frac{3}{2}|k|$ we estimate

$$
\begin{aligned}
\int_{|k'|>\frac{3}{2}|k|} \frac{1}{|k'|^{\alpha}} \frac{1}{|k-k'|^{\beta}} \frac{1}{|k|^2} dk' &\leq \int_{|k'|>\frac{3}{2}|k|} \frac{1}{(|k'|-|k|)^{\alpha}} \frac{1}{||k|-|k'||^{\beta}} \frac{1}{|k|^2} dk' \\
&= \int_{|k'|>\frac{3}{2}|k|} \frac{1}{(|k'|-|k|)^{\alpha}} \frac{1}{(|k'|-|k|)^{\beta}} \frac{1}{|k|^2} dk'
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{|k'|>\frac{3}{2}|k|} \frac{1}{(|k'|-|k|)^{\alpha+\beta}} \frac{1}{|k|^2} dk' = \int_{|k'|>\frac{3}{2}|k|} \frac{1}{(|k'|-|k|)^4} \frac{1}{|k|^2} dk' \\
&= \int_{\frac{3}{2}|k|}^{\infty} \frac{1}{(r-|k|)^4} \frac{1}{|k|^2} 4\pi r^2 dr = \frac{4\pi}{|k|^2} \int_{\frac{3}{2}|k|}^{\infty} \frac{1}{(r-|k|)^4} r^2 dr \\
&= \frac{4\pi}{|k|^2} \int_{\frac{1}{2}|k|}^{\infty} \frac{1}{r^4} (r+|k|)^2 dr = \frac{4\pi}{|k|^2} \int_{\frac{1}{2}|k|}^{\infty} \frac{1}{r^4} (r+|k|)^2 dr \\
&= \frac{4\pi}{|k|^2} \int_{\frac{1}{2}|k|}^{\infty} \frac{1}{r^4} (r^2 + 2r|k| + |k|^2) dr \\
&= \frac{4\pi}{|k|^2} \int_{\frac{1}{2}|k|}^{\infty} \left[\frac{1}{r^2} + 2|k| \frac{1}{r^3} + |k|^2 \frac{1}{r^4} \right] dr
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4\pi}{|k|^2}\left[\int_{\frac{1}{2}|k|}^{\infty} r^{-2}dr + 2|k|\int_{\frac{1}{2}|k|}^{\infty} r^{-3}dr + |k|^2\int_{\frac{1}{2}|k|}^{\infty} r^{-4}dr\right] \\
&= \frac{4\pi}{|k|^2}\left[\Big/_{\frac{1}{2}|k|}^{\infty} \frac{1}{-1}r^{-1} + 2|k|\Big/_{\frac{1}{2}|k|}^{\infty} \frac{1}{-2}r^{-2} + |k|^2\Big/_{\frac{1}{2}|k|}^{\infty} \frac{1}{-3}r^{-3}\right] \\
&= \frac{4\pi}{|k|^2}\left[(\frac{1}{2}|k|)^{-1} + 2|k|\frac{1}{2}(\frac{1}{2}|k|)^{-2} + |k|^2\frac{1}{3}(\frac{1}{2}|k|)^{-3}\right] \\
&= \frac{4\pi}{|k|^3}(2+4+\frac{8}{3}) \in \mathbb{R} \quad \forall k \neq 0.
\end{aligned}
$$

The remainder $\overline{B(0,\frac{3}{2}|k|)} \setminus (B(0,\frac{1}{2}|k|) \cup B(k,\frac{1}{2}|k|)$ of $\mathbb{R}^3$ is compact and the integrand continuous. This shows that the integrand has maximum. Positiveness of the integrand shows boundedness. Hence the integral exists over $\mathbb{R}^3$.

Assume $r' = r\sin(\theta)\cos(\theta)$. We have

$$
\begin{aligned}
I_1 &= \int_0^{\frac{\pi}{2}}\int_0^{\infty}\Big(\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)} \\
&\quad + \frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\Big)\frac{\sin(\theta)\cos(\theta)}{r^2\sin^2(\theta)\cos^2(\theta)+1}drd\theta \\
&= \int_0^{\frac{\pi}{2}}\Big(\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)} \\
&\quad + \frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\Big)\int_0^{\infty}\frac{\sin(\theta)\cos(\theta)}{r^2\sin^2(\theta)\cos^2(\theta)+1}drd\theta \\
&= \int_0^{\frac{\pi}{2}}\Big(\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)} \\
&\quad + \frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\Big)\int_0^{\infty}\frac{1}{r'^2+1}dr'd\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\infty}\frac{1}{r^2+1}dr\Big(\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)} \\
&\quad + \frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\Big)d\theta \\
&= \int_0^{\infty}\frac{1}{r^2+1}dr\int_0^{\frac{\pi}{2}}\Big(\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)} \\
&\quad + \frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\Big)d\theta \\
&= \frac{\pi}{2}\frac{2(\operatorname{artanh}(\frac{|k'|^2}{\sqrt{|k'|^4+|k-k'|^4}}) + \operatorname{artanh}(\frac{|k-k'|^2}{\sqrt{|k'|^4+|k-k'|^4}}))}{\sqrt{|k'|^4+|k-k'|^4}}
\end{aligned}
$$

whenever $k' \neq 0$ and $k' \neq k$. Hence

$$
\begin{aligned}
I_2 &= \int_0^{\frac{\pi}{2}}\int_0^{\infty}\Big|\Big(\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)} \\
&\quad + \frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\Big)\frac{\sin(\theta)\cos(\theta)}{r^2\sin^2(\theta)\cos^2(\theta)+1}\Big|drd\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{2}}\int_0^{\infty}\Big(\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)} \\
&\quad +\frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\Big)\frac{\sin(\theta)\cos(\theta)}{r^2\sin^2(\theta)\cos^2(\theta)+1}drd\theta \\
&= \frac{\pi}{2}\frac{2(\operatorname{artanh}(\frac{|k'|^2}{\sqrt{|k'|^4+|k-k'|^4}})+\operatorname{artanh}(\frac{|k-k'|^2}{\sqrt{|k'|^4+|k-k'|^4}}))}{\sqrt{|k'|^4+|k-k'|^4}} \in \mathbb{R}
\end{aligned}
$$

whenever $k' \neq 0$ and $k' \neq k$. The function

$$
\begin{aligned}
f(\tau',\tau'') &= \Big(\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)}+\frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\Big) \\
&\quad \frac{\sin(\theta)\cos(\theta)}{r^2\sin^2(\theta)\cos^2(\theta)+1}
\end{aligned}
$$

is clearly continuous and hence measurable. Sets $\mathbb{R}$ and $(0,\frac{\pi}{2})$ are both σ-compact sets because they are countable unions of compact sets. Hence they are also σ-finite measure spaces with Lebesgue measure. We estimate applying Fubini's theorem (Theorem 8.8 in [11]) to obtain

$$
\begin{aligned}
I &= \int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^{\infty}\int_0^{\infty}\frac{1}{|k'|^2\tau'+|k-k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1} \\
&\quad d\tau' d\tau'' dk' \qquad (4.13) \\
&= \frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^{\infty}\int_0^{\infty}\frac{1}{|k'|^2\tau'+|k-k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1} \\
&\quad d\tau' d\tau'' dk' \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^{\infty}\int_0^{\infty}\frac{1}{|k'|^2\tau'+|k-k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1} \\
&\quad d\tau' d\tau'' dk' \\
&= \frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^{\infty}\int_0^{\infty}\frac{1}{|k'|^2\tau'+|k-k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1} \\
&\quad d\tau' d\tau'' dk' \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k-k'|^2}\frac{1}{|k'|^2}\int_0^{\infty}\int_0^{\infty}\frac{1}{|k-k'|^2\tau'+|k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1} \\
&\quad d\tau' d\tau'' dk' \\
&= \frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^{\infty}\int_0^{\infty}\frac{1}{|k'|^2\tau'+|k-k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1} \\
&\quad d\tau' d\tau'' dk' \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^{\infty}\int_0^{\infty}\frac{1}{|k-k'|^2\tau'+|k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1} \\
&\quad d\tau' d\tau'' dk' \\
&= \frac{1}{2}\Big[\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^{\infty}\int_0^{\infty}\frac{1}{|k'|^2\tau'+|k-k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1} \\
&\quad d\tau' d\tau'' dk' \\
&\quad +\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^{\infty}\int_0^{\infty}\frac{1}{|k-k'|^2\tau'+|k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1} \\
&\quad d\tau' d\tau'' dk'\Big]
\end{aligned}
$$

$$= \frac{1}{2}\int_{\mathbb{R}^3}\left[\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^\infty\int_0^\infty \frac{1}{|k'|^2\tau'+|k-k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}\right.$$
$$d\tau' d\tau''$$
$$+\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^\infty\int_0^\infty \frac{1}{|k-k'|^2\tau'+|k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}$$
$$\left. d\tau' d\tau''\right] dk'$$

$$= \frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\left[\int_0^\infty\int_0^\infty \frac{1}{|k'|^2\tau'+|k-k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}\right.$$
$$d\tau' d\tau''$$
$$\left.+\int_0^\infty\int_0^\infty \frac{1}{|k-k'|^2\tau'+|k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau' d\tau''\right] dk'$$

$$= \frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^\infty\left[\int_0^\infty \frac{1}{|k'|^2\tau'+|k-k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}\right.$$
$$d\tau'$$
$$\left.+\int_0^\infty \frac{1}{|k-k'|^2\tau'+|k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau'\right] d\tau'' dk'$$

$$= \frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^\infty\int_0^\infty\left[\frac{1}{|k'|^2\tau'+|k-k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}\right.$$
$$\left.+\frac{1}{|k-k'|^2\tau'+|k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}\right] d\tau' d\tau'' dk'$$

$$= \frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^\infty\int_0^\infty\left[\frac{1}{|k'|^2\tau'+|k-k'|^2\tau''}\right.$$
$$\left.+\frac{1}{|k-k'|^2\tau'+|k'|^2\tau''}\right]\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau' d\tau'' dk'$$

$$= \frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\left[\frac{1}{|k'|^2 r\cos(\theta)+|k-k'|^2 r\sin(\theta)}\right.$$
$$\left.+\frac{1}{|k-k'|^2 r\cos(\theta)+|k'|^2 r\sin(\theta)}\right]\frac{r\sin(\theta)}{r^2\sin^2(\theta)+1}\frac{r\cos(\theta)}{r^2\cos^2(\theta)+1}$$
$$r d\theta dr dk'$$

$$= \frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\left[\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)}\right.$$
$$\left.+\frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\right]$$
$$\frac{r^2\sin(\theta)\cos(\theta)}{r^2\sin^2(\theta)r^2\cos^2(\theta)+r^2\sin^2(\theta)+r^2\cos^2(\theta)+1}d\theta dr dk'$$

$$\leq \frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\left[\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)}\right.$$
$$\left.+\frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\right]$$
$$\frac{r^2\sin(\theta)\cos(\theta)}{r^2\sin^2(\theta)r^2\cos^2(\theta)+r^2\sin^2(\theta)+r^2\cos^2(\theta)}d\theta dr dk'$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^3} \frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\left[\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)}\right. \\
&\quad \left.+\frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\right] \\
&\quad \frac{\sin(\theta)\cos(\theta)}{\sin^2(\theta)r^2\cos^2(\theta)+\sin^2(\theta)+\cos^2(\theta)}d\theta dr dk' \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\left[\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)}\right. \\
&\quad \left.+\frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\right]\frac{\sin(\theta)\cos(\theta)}{r^2\sin^2(\theta)\cos^2(\theta)+1}d\theta dr dk' \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\int_0^{\frac{\pi}{2}}\int_0^\infty\left[\frac{1}{|k'|^2\cos(\theta)+|k-k'|^2\sin(\theta)}\right. \\
&\quad \left.+\frac{1}{|k-k'|^2\cos(\theta)+|k'|^2\sin(\theta)}\right]\frac{\sin(\theta)\cos(\theta)}{r^2\sin^2(\theta)\cos^2(\theta)+1}dr d\theta dk' \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\frac{\pi}{2}\frac{2\left[\operatorname{artanh}(\frac{|k'|^2}{\sqrt{|k'|^4+|k-k'|^4}})+\operatorname{artanh}(\frac{|k-k'|^2}{\sqrt{|k'|^4+|k-k'|^4}})\right]}{\sqrt{|k'|^4+|k-k'|^4}} \\
&\quad dk' \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\frac{\pi}{2}\frac{1}{\sqrt{\frac{1}{16}|k|^4}}2\left[2+4\frac{\sqrt{|k'|}}{\sqrt{|k-k'|}}+2+4\frac{\sqrt{|k-k'|}}{\sqrt{|k'|}}\right]dk' \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\frac{\pi}{2}\frac{4}{|k|^2}2\cdot 4\left[1+\frac{\sqrt{|k'|}}{\sqrt{|k-k'|}}+\frac{\sqrt{|k-k'|}}{\sqrt{|k'|}}\right]dk' \\
&= 8\pi\int_{\mathbb{R}^3} \frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\frac{1}{|k|^2}\left[1+\frac{\sqrt{|k'|}}{\sqrt{|k-k'|}}+\frac{\sqrt{|k-k'|}}{\sqrt{|k'|}}\right]dk' \\
&= 8\pi\int_{\mathbb{R}^3} \frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\frac{1}{|k|^2}+\frac{1}{|k'|^{\frac{3}{2}}}\frac{1}{|k-k'|^{\frac{5}{2}}}\frac{1}{|k|^2}+\frac{1}{|k'|^{\frac{5}{2}}}\frac{1}{|k-k'|^{\frac{3}{2}}}dk' \\
&= 8\pi\int_{\mathbb{R}^3} \frac{1}{|k'|^2}\frac{1}{|k-k'|^2}\frac{1}{|k|^2}dk'+8\pi\int_{\mathbb{R}^3}\frac{1}{|k'|^{\frac{3}{2}}}\frac{1}{|k-k'|^{\frac{5}{2}}}\frac{1}{|k|^2}dk' \\
&\quad +8\pi\int_{\mathbb{R}^3}\frac{1}{|k'|^{\frac{5}{2}}}\frac{1}{|k-k'|^{\frac{3}{2}}}\frac{1}{|k|^2}dk' \in \mathbb{R} \quad \forall k \neq 0.
\end{aligned}
$$

This shows that the expression (4.13) exists for all $k \neq 0$. We see that it is constant over rotations $\acute{k} = Ak$ and hence has a constant value over any sphere. This is a consequence of the change of variables $k' = Ak''$, linearity of A, the equation $|Ak| = |k|$ and the equation $\det(A) = 1$. We now calculate

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\frac{1}{|k'|^2}\frac{1}{|\beta k-k'|^2}\int_0^\infty\int_0^\infty\frac{1}{|k'|^2\tau'+|\beta k-k'|^2\tau''}\frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1} \\
&\quad d\tau' d\tau'' dk'|\beta k|^3 \\
&= \int_{\mathbb{R}^3}\frac{1}{|\beta k'|^2}\frac{1}{|\beta k-\beta k'|^2}\int_0^\infty\int_0^\infty\frac{1}{|\beta k'|^2\tau'+|\beta k-\beta k'|^2\tau''} \\
&\quad \frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau' d\tau''\beta^3 dk'|\beta k|^3
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} \frac{1}{|k'|^2} \frac{1}{|k-k'|^2} \int_0^\infty \int_0^\infty \frac{1}{|k'|^2\tau' + |k-k'|^2\tau''} \frac{\tau'}{\tau'^2+1} \\
&\quad \frac{\tau''}{\tau''^2+1} d\tau' d\tau'' dk' |k|^3 \\
&= C \quad \forall \beta > 0
\end{aligned}
$$

to obtain that (4.13) is constant C for every $k \neq 0$. We can now show boundedness of T. Assume that u is bounded. We have

$$
\begin{aligned}
|Tu(k,\omega)| &= \Bigg| -\hat{\Pi}^S(k) \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{u_m(k',\omega')}{|k'|^2(i\omega' + \nu|k'|^2)} \\
&\quad \frac{u_n(k-k',\omega-\omega')}{|k-k'|^2(i(\omega-\omega') + \nu|k-k'|^2)} d\omega' dk' i k_m e_n |k|^2 \Bigg| \\
&\leq \left| -\hat{\Pi}^S(k) \right| \Bigg| \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{u_m(k',\omega')}{|k'|^2(i\omega' + \nu|k'|^2)} \\
&\quad \frac{u_n(k-k',\omega-\omega')}{|k-k'|^2(i(\omega-\omega') + \nu|k-k'|^2)} d\omega' dk' i k_m e_n |k|^2 \Bigg| \\
&\leq \left| -\hat{\Pi}^S(k) \right| \Bigg| \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{u_m(k',\omega')}{|k'|^2(i\omega' + \nu|k'|^2)} \\
&\quad \frac{u_n(k-k',\omega-\omega')}{|k-k'|^2(i(\omega-\omega') + \nu|k-k'|^2)} d\omega' dk' i k_m e_n \Bigg| |k|^2 \\
&\leq \left| -\hat{\Pi}^S(k) \right| \sum_{n=1}^{3} \Bigg| \sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{u_m(k',\omega')}{|k'|^2(i\omega' + \nu|k'|^2)} \\
&\quad \frac{u_n(k-k',\omega-\omega')}{|k-k'|^2(i(\omega-\omega') + \nu|k-k'|^2)} d\omega' dk' i k_m \Bigg| |k|^2 \\
&\leq \left| -\hat{\Pi}^S(k) \right| \sum_{n=1}^{3}\sum_{m=1}^{3} \Bigg| \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{u_m(k',\omega')}{|k'|^2(i\omega' + \nu|k'|^2)} \\
&\quad \frac{u_n(k-k',\omega-\omega')}{|k-k'|^2(i(\omega-\omega') + \nu|k-k'|^2)} d\omega' dk' i k_m \Bigg| |k|^2 \\
&\leq \left| -\hat{\Pi}^S(k) \right| \sum_{n=1}^{3}\sum_{m=1}^{3} \Bigg| \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{u_m(k',\omega')}{|k'|^2(i\omega' + \nu|k'|^2)} \\
&\quad \frac{u_n(k-k',\omega-\omega')}{|k-k'|^2(i(\omega-\omega') + \nu|k-k'|^2)} d\omega' dk' \Bigg| |k| |k|^2 \\
&\leq \left| -\hat{\Pi}^S(k) \right| \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Bigg| \frac{u_m(k',\omega')}{|k'|^2(i\omega' + \nu|k'|^2)} \\
&\quad \frac{u_n(k-k',\omega-\omega')}{|k-k'|^2(i(\omega-\omega') + \nu|k-k'|^2)} \Bigg| d\omega' dk' |k| |k|^2
\end{aligned}
$$

$$
\begin{aligned}
\leq & \left|-\hat{\Pi}^S(k)\right| \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|u(k',\omega')|}{|k'|^2|i\omega'+\nu|k'|^2|} \\
& \frac{|u(k-k',\omega-\omega')|}{|k-k'|^2|i(\omega-\omega')+\nu|k-k'|^2|} d\omega' dk' |k||k|^2 \\
\leq & \left|-\hat{\Pi}^S(k)\right| \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{||u||}{|k'|^2|i\omega'+\nu|k'|^2|} \\
& \frac{||u||}{|k-k'|^2|i(\omega-\omega')+\nu|k-k'|^2|} d\omega' dk' |k||k|^2 \\
= & \left|-\hat{\Pi}^S(k)\right| 9 \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{||u||}{|k'|^2|i\omega'+\nu|k'|^2|} \\
& \frac{||u||}{|k-k'|^2|i(\omega-\omega')+\nu|k-k'|^2|} d\omega' dk' |k||k|^2 \\
\leq & \left|-\hat{\Pi}^S(k)\right| 9||u||^2 \int_{\mathbb{R}^3} \frac{1}{|k'|^2} \frac{1}{|k-k'|^2} \\
& \int_{\mathbb{R}} \frac{1}{|i\omega'+\nu|k'|^2|} \frac{1}{|i(\omega-\omega')+\nu|k-k'|^2|} d\omega' \\
& dk'|k|^3 \\
\leq & \left|-\hat{\Pi}^S(k)\right| 9||u||^2 \int_{\mathbb{R}^3} \frac{1}{|k'|^2} \frac{1}{|k-k'|^2} \frac{16}{\pi} \\
& \int_0^\infty \int_0^\infty \frac{1}{\tau'\nu|k'|^2+\tau''\nu|k-k'|^2} \\
& \frac{\tau'}{\tau'^2+1} \frac{\tau''}{\tau''^2+1} d\tau' d\tau'' dk' |k|^3 \\
\leq & \left|-\hat{\Pi}^S(k)\right| 9||u||^2 \int_{\mathbb{R}^3} \frac{1}{|k'|^2} \frac{1}{|k-k'|^2} \frac{16}{\pi\nu} \\
& \int_0^\infty \int_0^\infty \frac{1}{\tau'|k'|^2+\tau''|k-k'|^2} \\
& \frac{\tau'}{\tau'^2+1} \frac{\tau''}{\tau''^2+1} d\tau' d\tau'' dk' |k|^3 \\
\leq & \left|\hat{\Pi}^S(k)\right| 9||u||^2 \frac{16}{\pi\nu} \int_{\mathbb{R}^3} \frac{1}{|k'|^2} \frac{1}{|k-k'|^2} \\
& \int_0^\infty \int_0^\infty \frac{1}{\tau'\nu|k'|^2+\tau''\nu|k-k'|^2} \\
& \frac{\tau'}{\tau'^2+1} \frac{\tau''}{\tau''^2+1} d\tau' d\tau'' dk' |k|^3 \\
\leq & 2\cdot 9||u||^2 \frac{16}{\pi\nu} C = \frac{288\,C}{\pi\nu}||u||^2,
\end{aligned}
$$

that implies

$$
\begin{aligned}
||Tu|| &= \sup_{k\in S,\omega\in\mathbb{R}} |Tu(k,\omega)| \leq \sup_{k\in S,\omega\in\mathbb{R}} \frac{288\,C}{\pi\nu}||u||^2 = \frac{288\,C}{\pi\nu}||u||^2, \\
||Tu|| &\leq \frac{288\,C}{\pi\nu}||u||^2. \qquad (4.14)
\end{aligned}
$$

Also

$$||g|| = \sup_{k\in S,\omega\in\mathbb{R}} |g(k,\omega)| = \sup_{k\in H,\omega\in\mathbb{R}} |f(k)|k|^2| \in \mathbb{R}.$$

This is because $f \in \mathscr{S}_3$. The estimate (4.14) shows boundedness of T. Fourier inversion identity was applied.

4.4 L^1 - estimate

We know that convolution has the estimate

$$||f * g||_1 \leq ||f||_1||g||_1. \tag{4.15}$$

To show the estimate (4.15), we assume that functions $f : \mathbb{R}^n \to \mathbb{C}$ and $g : \mathbb{R}^n \to \mathbb{C}$ are absolutely Riemann-integrable and $f * g$ exists. We calculate

$$\begin{aligned}
||f||_1||g||_1 &= ||g||_1||f||_1 = \int_{\mathbb{R}^n} |g(t)|dt \int_{\mathbb{R}^n} |f(\tau)|d\tau = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |g(t)|dt|f(\tau)|d\tau \\
&= \int_{\mathbb{R}^n} |f(\tau)| \int_{\mathbb{R}^n} |g(t)|dtd\tau = \int_{\mathbb{R}^n} |f(\tau)| \int_{\mathbb{R}^n} |g(t-\tau)|dtd\tau \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(\tau)||g(t-\tau)|dtd\tau = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(\tau)g(t-\tau)|dtd\tau,
\end{aligned}$$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(\tau)||g(t-\tau)|dtd\tau = ||f||_1||g||_1 \in \mathbb{R}.$$

Sets $\mathbb{R}^n$ and $\mathbb{R}^n$ are σ-compact. The construction is similar to $\mathbb{Z}^n$ in both cases. Hence they are σ-finite measure spaces with Lebesgue measure. Integrand is measurable as a product of measurable functions by Theorem 1.9 in [11]. We apply Fubini's theorem (Theorem 8.8 in [11]) to obtain

$$\begin{aligned}
||f * g||_1 &= \int_{\mathbb{R}^n} |f * g(t)|dt = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(\tau)g(t-\tau)d\tau \right| dt \\
&\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(\tau)g(t-\tau)|d\tau dt \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(\tau)g(t-\tau)|dtd\tau = ||f||_1||g||_1.
\end{aligned}$$

We know that the class L^1 is complete and its elements have continuous inverse Fourier transforms. In addition, Fourier-domain L^1-Cauchy sequence is uniformly convergent in spacetime and hence has a pointwise limit. Hence it is wise to introduce a formulation for our problem for functions $\tilde{u}(k, \omega - ia)$ that are absolutely Riemann-integrable for every $a \in \mathbb{R} \setminus \mathbb{R}_-$. We start from the equation (4.2) and calculate

$$\begin{aligned}
u(k,\omega) &= -\hat{\Pi}^S(k)\; u * u(k,\omega)\; \frac{ik}{i\omega + \nu|k|^2} + \frac{f(k)}{i\omega + \nu|k|^2} \\
&= -\hat{\Pi}^S(k) \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} u_m(k',\omega')u_n(k-k',\omega-\omega')d\omega'dk' \\
&\quad \left[\frac{ik}{i\omega + \nu|k|^2}\right]_m e_n + \frac{f(k)}{i\omega + \nu|k|^2}
\end{aligned}$$

$$= \quad -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} u_m(k',\omega')u_n(k-k',\omega-\omega')d\omega' dk'$$
$$\frac{ik_m e_n}{i\omega+\nu|k|^2}+\frac{f(k)}{i\omega+\nu|k|^2}.$$

We make the change of variables

$$\tilde{u}(k,\omega)=u(k,\omega)-\frac{f(k)}{i\omega+\nu|k|^2}. \tag{4.16}$$

Boundedness and continuity of $\tilde{u}$ follows from properties of u handled in Section 4.1 and the change of variables (4.16). We obtain formulation

$$\begin{aligned}
u(k,\omega) \quad = \quad & -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} u_m(k',\omega')u_n(k-k',\omega-\omega')d\omega' dk' \\
& \frac{ik_m e_n}{i\omega+\nu|k|^2} \\
& -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{f_m(k')}{i\omega'+\nu|k'|^2}u_n(k-k',\omega-\omega')d\omega' dk' \\
& \frac{ik_m e_n}{i\omega+\nu|k|^2} \\
& -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} u_m(k',\omega')\frac{f_n(k-k')}{i(\omega-\omega')+\nu|k-k'|^2}d\omega' dk' \\
& \frac{ik_m e_n}{i\omega+\nu|k|^2} \\
& -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{f_m(k')}{i\omega'+\nu|k'|^2}\frac{f_n(k-k')}{i(\omega-\omega')+\nu|k-k'|^2}d\omega' \\
& dk'\frac{ik_m e_n}{i\omega+\nu|k|^2},
\end{aligned} \tag{4.17}$$

where we have omitted tilde for notational convenience. Note that $\frac{f(k)}{i\omega+\nu|k|^2}$ is the Fourier transform of the solution of the heat equation with initial value field v_0. Hence substraction by $\frac{f(k)}{i\omega+\nu|k|^2}$ causes initial value fields to cancel, which makes the difference field continuous in spacetime. We define

$$\begin{aligned}
T_1u(k,\omega) \quad = \quad & -\hat{\Pi}^S(k)\sum_{m=1}^{3}\sum_{n=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} u_m(k',\omega')u_n(k-k',\omega-\omega')d\omega' dk' \\
& \frac{ik_m e_n}{i\omega+\nu|k|^2} \\
T_2u(k,\omega) \quad = \quad & -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{f_m(k')}{i\omega'+\nu|k'|^2}u(k-k',\omega-\omega')d\omega' dk' \\
& \frac{ik_m e_n}{i\omega'+\nu|k'|^2}
\end{aligned}$$

$$
\begin{aligned}
T_3u(k,\omega) &= -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} u_m(k-k',\omega-\omega')\frac{f_n(k')}{i\omega'+\nu|k'|^2}d\omega' dk' \\
&\quad \frac{ik_m e_n}{i\omega+\nu|k|^2} \\
g(k,\omega) &= -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{f_m(k')}{i\omega'+\nu|k'|^2}\frac{f_n(k-k')}{i(\omega-\omega')+\nu|k-k'|^2}d\omega' \\
&\quad dk'\frac{ik_m e_n}{i\omega+\nu|k|^2} \\
T &= T_1+T_2+T_3
\end{aligned}
$$

and obtain the formulation

$$
u(k,\omega) = Tu(k,\omega)+g(k,\omega), \quad k\neq 0, \quad \omega\in\mathbb{R} \tag{4.18}
$$

for (4.17).

We define α-norm

$$
|||u|||_\alpha = ||\check{\tau}_{i\alpha^2}u||_1, \tag{4.19}
$$

where $\alpha\in\mathbb{R}_+$. We calculate

$$
|||u|||_\alpha = ||\check{\tau}_{i\alpha^2}u||_1 \geq 0.
$$

This shows positiveness of α-norm. We calculate

$$
|||au|||_\alpha = ||\check{\tau}_{i\alpha^2}au||_1 = ||a\check{\tau}_{i\alpha^2}u||_1 = |a|||\check{\tau}_{i\alpha^2}u||_1 = |a|||u|||_\alpha.
$$

This shows homogeneity of α-norm. We calculate

$$
\begin{aligned}
|||u_1+u_2|||_\alpha &= ||\check{\tau}_{i\alpha^2}(u_1+u_2)||_1 = ||\check{\tau}_{i\alpha^2}u_1+\check{\tau}_{i\alpha^2}u_2||_1 \\
&\leq ||\check{\tau}_{i\alpha^2}u_1||+||\check{\tau}_{i\alpha^2}u_2||_1 \\
&= |||u_1|||_\alpha+|||u_2|||_\alpha.
\end{aligned}
$$

This shows the triangle inequality for α-norm. Assume $u=0$. We calculate

$$
|||u|||_\alpha = ||\check{\tau}_{i\alpha^2}u||_1 = ||\check{\tau}_{i\alpha^2}0||_1 = ||0||_1 = 0.
$$

Assume $|||u|||_\alpha = 0$. Recall that $u(k,\omega)$ was shown to be analytic on the lower complex plane $\mathrm{Im}(\omega)<0$ for every $k\neq 0$ and continuous to the boundary $\mathbb{R}$. In addition u was assumed to be continuous on Z. This implies $\check{\tau}_{i\alpha^2}u(k,\omega)=0$ for every $(k,\omega)\in S\times\mathbb{R}$ and hence $u(k,\omega)=0$ on the line $\mathrm{Im}(\omega)=-\alpha^2$ for every $k\neq 0$. Analyticity and continuity of u to the boundary $\mathbb{R}$ now imply $u=0$ (see respective proof in Section 4.1). Hence we have

$$
u=0 \quad \Leftrightarrow \quad |||u|||_\alpha = 0.
$$

This shows that α-norm satisfies norm axioms. We have

$$
|||u|||_\alpha = \int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k,\omega-i\alpha^2)|d\omega dk.
$$

The latter integral in the change of order of integration is absolutely integrable by the rest of the estimate. Hence we apply Theorem 8.8 in [11] to obtain

$$\begin{aligned}
|||T_1 u|||_\alpha &= \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Bigg| -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} u_m(k',\omega') \\
&\quad u_n(k-k',\omega-i\alpha^2-\omega')d\omega' dk' \frac{ik_m e_n}{i\omega+\alpha^2+\nu|k|^2}\Bigg| d\omega dk \\
&\le \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Big| -\hat{\Pi}^S(k)\Big| \Bigg|\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} u_m(k',\omega') \\
&\quad u_n(k-k',\omega-i\alpha^2-\omega')d\omega' dk' \frac{ik_m e_n}{i\omega+\alpha^2+\nu|k|^2}\Bigg| d\omega dk \\
&\le \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Big| -\hat{\Pi}^S(k)\Big| \sum_{n=1}^{3}\Bigg|\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} u_m(k',\omega') \\
&\quad u_n(k-k',\omega-i\alpha^2-\omega')d\omega' dk' \frac{ik_m e_n}{i\omega+\alpha^2+\nu|k|^2}\Bigg| d\omega dk \\
&\le \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Big| -\hat{\Pi}^S(k)\Big| \sum_{n=1}^{3}\sum_{m=1}^{3}\Bigg|\int_{\mathbb{R}^3}\int_{\mathbb{R}} u_m(k',\omega') \\
&\quad u_n(k-k',\omega-i\alpha^2-\omega')d\omega' dk' \frac{ik_m e_n}{i\omega+\alpha^2+\nu|k|^2}\Bigg| d\omega dk \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Big| -\hat{\Pi}^S(k)\Big| \sum_{n=1}^{3}\sum_{m=1}^{3}\Bigg|\int_{\mathbb{R}^3}\int_{\mathbb{R}} u_m(k',\omega') \\
&\quad u_n(k-k',\omega-i\alpha^2-\omega')d\omega' dk' \Bigg|\Bigg| \frac{ik_m e_n}{i\omega+\alpha^2+\nu|k|^2}\Bigg| d\omega dk \\
&\le \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Big| -\hat{\Pi}^S(k)\Big| \sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} |u_m(k',\omega') \\
&\quad u_n(k-k',\omega-i\alpha^2-\omega')|d\omega' dk' \frac{|i||k_m||e_n|}{|i\omega+\alpha^2+\nu|k|^2|} d\omega dk \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Big| -\hat{\Pi}^S(k)\Big| \sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} |u_m(k',\omega')| \\
&\quad |u_n(k-k',\omega-i\alpha^2-\omega')|d\omega' dk' \frac{|k_m|}{|i\omega+\alpha^2+\nu|k|^2|} d\omega dk \\
&\le \int_{\mathbb{R}^3}\int_{\mathbb{R}} 2\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k',\omega')||u(k-k',\omega-i\alpha^2-\omega')|d\omega' dk' \\
&\quad \frac{|k|}{|\alpha^2+\nu|k|^2|} d\omega dk \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}} 2\cdot 9\int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k',\omega')||u(k-k',\omega-i\alpha^2-\omega')|d\omega' dk' \\
&\quad \frac{|k|}{\alpha^2+\nu|k|^2} d\omega dk
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}} 18 \int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k',\omega')||u(k-k',\omega-i\alpha^2-\omega')|d\omega' dk' \\
&\quad \frac{1}{\alpha\sqrt{\nu}} d\omega dk \\
&= \frac{18}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k',\omega')||u(k-k',\omega-i\alpha^2-\omega')|d\omega' dk' \\
&\quad d\omega dk \\
&= \frac{18}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k',\omega')||u(k-k',\omega-i\alpha^2-\omega')|d\omega dk \\
&\quad d\omega' dk' \\
&= \frac{18}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k',\omega')| \int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k-k',\omega-i\alpha^2-\omega')|d\omega dk \\
&\quad d\omega' dk' \\
&= \frac{18}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k',\omega')| \int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k,\omega-i\alpha^2)|d\omega dk d\omega' dk' \\
&= \frac{18}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k,\omega-i\alpha^2)|d\omega dk |u(k',\omega')|d\omega' dk' \\
&= \frac{18}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k,\omega-i\alpha^2)|d\omega dk \int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k',\omega')|d\omega' dk' \\
&= \frac{18}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k',\omega')|d\omega' dk' \int_{\mathbb{R}^3}\int_{\mathbb{R}} |u(k,\omega-i\alpha^2)|d\omega dk \\
&= \frac{18}{\alpha\sqrt{\nu}} ||u||_1 |||u|||_\alpha .
\end{aligned}
$$

The latter integral in the change of order of integration is again absolutely integrable by the rest of the estimate and we apply Theorem 8.8 in [11] to obtain

$$
\begin{aligned}
I &= \int_{\mathbb{R}^3}\int_{\mathbb{R}} 2\cdot 9 \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|f(k')|}{|i\omega'+\nu|k'|^2|} \\
&\quad |u(k-k',\omega-i\alpha^2-\omega')|d\omega' dk' \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|} d\omega dk \\
&= 2\cdot 9 \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|f(k')|}{|i\omega'+\nu|k'|^2|} \\
&\quad |u(k-k',\omega-i\alpha^2-\omega')|d\omega' dk' \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|} d\omega dk \\
&= 18 \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|f(k-k')|}{|i(\omega-\omega')+\nu|k-k'|^2|} \\
&\quad |u(k',\omega'-i\alpha^2)|d\omega' dk' \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|} d\omega dk \\
&= 18 \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|f(k-k')|}{|i(\omega-\omega')+\nu|k-k'|^2|} \\
&\quad |u(k',\omega'-i\alpha^2)|d\omega' dk' d\omega dk \\
&= 18 \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|} \frac{|f(k-k')|}{|i(\omega-\omega')+\nu|k-k'|^2|} \\
&\quad |u(k',\omega'-i\alpha^2)|d\omega' dk' d\omega dk
\end{aligned}
$$

$$
\begin{aligned}
&= 18\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{|f(k-k')|}{|i(\omega-\omega')+\nu|k-k'|^2|} \\
&\quad |u(k',\omega'-i\alpha^2)|\frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega' dk' d\omega dk \\
&= 18\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{|f(k-k')|}{|i(\omega-\omega')+\nu|k-k'|^2|} \\
&\quad |u(k',\omega'-i\alpha^2)|\frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk d\omega' dk' \\
&= 18\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}|u(k',\omega'-i\alpha^2)|\frac{|f(k-k')|}{|i(\omega-\omega')+\nu|k-k'|^2|} \\
&\quad \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk d\omega' dk' \\
&= 18\int_{\mathbb{R}^3}\int_{\mathbb{R}}|u(k',\omega'-i\alpha^2)|\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{|f(k-k')|}{|i(\omega-\omega')+\nu|k-k'|^2|} \\
&\quad \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk d\omega' dk' \\
&= 18\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{|f(k-k')|}{|i(\omega-\omega')+\nu|k-k'|^2|} \\
&\quad \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&= 18\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k-k')||k|\int_{\mathbb{R}}\frac{1}{|i(\omega-\omega')+\nu|k-k'|^2|} \\
&\quad \frac{1}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&= 18\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k-k')||k|\int_{\mathbb{R}}\frac{1}{|-i(\omega'-\omega)+\nu|k'-k|^2|} \\
&\quad \frac{1}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&= 18\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k-k')||k|\int_{\mathbb{R}}\frac{1}{|i(\omega'-\omega)+\nu|k'-k|^2|} \\
&\quad \frac{1}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&= 18\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k-k')||k|\int_{\mathbb{R}}\frac{1}{|i\omega+\alpha^2+\nu|k|^2|} \\
&\quad \frac{1}{|i(\omega'-\omega)+\nu|k'-k|^2|}d\omega dk|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&\leq 18\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k-k')||k| \\
&\quad \frac{16}{\pi}\int_0^\infty\int_0^\infty\frac{1}{\tau'\nu|k-k'|^2+\tau''(\alpha^2+\nu|k|^2)} \\
&\quad \frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau' d\tau'' dk|u(k',\omega'-i\alpha^2)|d\omega' dk'
\end{aligned}
$$

$$
\begin{aligned}
&\leq 18\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k-k')||k| \\
&\quad \frac{16}{\pi}\int_0^\infty\int_0^\infty \frac{1}{\sqrt{\tau''\nu|k|^2}\sqrt{\tau'\nu|k-k'|^2+\tau''(\alpha^2+\nu|k|^2)}} \\
&\quad \frac{\tau'}{\tau'^2+1}\frac{\tau''}{\tau''^2+1}d\tau' d\tau'' dk|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&\leq \frac{18}{\sqrt{\nu}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k-k')| \\
&\quad \frac{16}{\pi}\int_0^\infty\int_0^\infty \frac{1}{\sqrt{\tau'\nu|k-k'|^2+\tau''(\alpha^2+\nu|k|^2)}} \\
&\quad \frac{\tau'}{\tau'^2+1}\frac{\tau''^{\frac{1}{2}}}{\tau''^2+1}d\tau' d\tau'' dk|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&\leq \frac{18}{\sqrt{\nu}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k-k')| \\
&\quad \frac{16}{\pi}\int_0^\infty\int_0^\infty \frac{1}{(\tau'\nu|k-k'|^2)^{\frac{1}{4}}(\tau''\alpha^2)^{\frac{1}{4}}} \\
&\quad \frac{\tau'}{\tau'^2+1}\frac{\tau''^{\frac{1}{2}}}{\tau''^2+1}d\tau' d\tau'' dk|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&= \frac{18}{\alpha^{\frac{1}{2}}\nu^{\frac{3}{4}}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k-k')||k-k'|^{-\frac{1}{2}} \\
&\quad \frac{16}{\pi}\int_0^\infty\int_0^\infty \frac{\tau'^{\frac{3}{4}}}{\tau'^2+1}\frac{\tau''^{\frac{1}{4}}}{\tau''^2+1}d\tau' d\tau'' dk|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&= \frac{18}{\alpha^{\frac{1}{2}}\nu^{\frac{3}{4}}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k-k')||k-k'|^{-\frac{1}{2}} \\
&\quad \frac{16}{\pi}\int_0^\infty \frac{\tau'^{\frac{3}{4}}}{\tau'^2+1}d\tau'\int_0^\infty\frac{\tau''^{\frac{1}{4}}}{\tau''^2+1}d\tau'' dk|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&= \frac{18}{\alpha^{\frac{1}{2}}\nu^{\frac{3}{4}}}\frac{16}{\pi}\int_0^\infty \frac{\tau'^{\frac{3}{4}}}{\tau'^2+1}d\tau'\int_0^\infty\frac{\tau''^{\frac{1}{4}}}{\tau''^2+1}d\tau'' \\
&\quad \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k-k')||k-k'|^{-\frac{1}{2}}dk|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&= \frac{18}{\alpha^{\frac{1}{2}}\nu^{\frac{3}{4}}}\frac{16}{\pi}\int_0^\infty \frac{\tau'^{\frac{3}{4}}}{\tau'^2+1}d\tau'\int_0^\infty\frac{\tau''^{\frac{1}{4}}}{\tau''^2+1}d\tau'' \\
&\quad \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k)||k|^{-\frac{1}{2}}dk|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&= \frac{18}{\alpha^{\frac{1}{2}}\nu^{\frac{3}{4}}}\frac{16}{\pi}\int_0^\infty \frac{\tau'^{\frac{3}{4}}}{\tau'^2+1}d\tau'\int_0^\infty\frac{\tau''^{\frac{1}{4}}}{\tau''^2+1}d\tau'' \\
&\quad \int_{\mathbb{R}^3}|f(k)||k|^{-\frac{1}{2}}dk\int_{\mathbb{R}^3}\int_{\mathbb{R}}|u(k',\omega'-i\alpha^2)|d\omega' dk' \\
&= \frac{C_1}{\alpha^{\frac{1}{2}}\nu^{\frac{3}{4}}}\int_{\mathbb{R}^3}|f(k)||k|^{-\frac{1}{2}}dk\,|||u|||_\alpha.
\end{aligned}
$$

Hence we can estimate

$$
\begin{aligned}
|||T_2u|||_\alpha &= \int_{\mathbb{R}^3}\int_{\mathbb{R}}\Bigg| -\hat{\Pi}^S(k)\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{f_m(k')}{i\omega'+\nu|k'|^2} \\
&\quad u_n(k-k',\omega-i\alpha^2-\omega')d\omega'dk'\frac{ik_me_n}{i\omega+\alpha^2+\nu|k|^2}\Bigg|d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}}\Big| -\hat{\Pi}^S(k)\Big|\Bigg|\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{f_m(k')}{i\omega'+\nu|k'|^2} \\
&\quad u_n(k-k',\omega-i\alpha^2-\omega')d\omega'dk'\frac{ik_me_n}{i\omega+\alpha^2+\nu|k|^2}\Bigg|d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}}\Big| -\hat{\Pi}^S(k)\Big|\sum_{n=1}^{3}\Bigg|\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{f_m(k')}{i\omega'+\nu|k'|^2} \\
&\quad u_n(k-k',\omega-i\alpha^2-\omega')d\omega'dk'\frac{ik_me_n}{i\omega+\alpha^2+\nu|k|^2}\Bigg|d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}}\Big| -\hat{\Pi}^S(k)\Big|\sum_{n=1}^{3}\sum_{m=1}^{3}\Bigg|\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{f_m(k')}{i\omega'+\nu|k'|^2} \\
&\quad u_n(k-k',\omega-i\alpha^2-\omega')d\omega'dk'\frac{ik_me_n}{i\omega+\alpha^2+\nu|k|^2}\Bigg|d\omega dk \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}}\Big| -\hat{\Pi}^S(k)\Big|\sum_{n=1}^{3}\sum_{m=1}^{3}\Bigg|\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{f_m(k')}{i\omega'+\nu|k'|^2} \\
&\quad u_n(k-k',\omega-i\alpha^2-\omega')d\omega'dk'\Bigg|\Bigg|\frac{ik_me_n}{i\omega+\alpha^2+\nu|k|^2}\Bigg|d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}}\Big| -\hat{\Pi}^S(k)\Big|\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\Bigg|\frac{f_m(k')}{i\omega'+\nu|k'|^2} \\
&\quad u_n(k-k',\omega-i\alpha^2-\omega')\Bigg|d\omega'dk'\frac{|i||k_m||e_n|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}}\Big| -\hat{\Pi}^S(k)\Big|\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{|f_m(k')|}{|i\omega'+\nu|k'|^2|} \\
&\quad |u_n(k-k',\omega-i\alpha^2-\omega')|d\omega'dk'\frac{|k_m|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}}2\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{|f_m(k')|}{|i\omega'+\nu|k'|^2|} \\
&\quad |u_n(k-k',\omega-i\alpha^2-\omega')|d\omega'dk'\frac{|k_m|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}}2\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{|f(k')|}{|i\omega'+\nu|k'|^2|} \\
&\quad |u(k-k',\omega-i\alpha^2-\omega')|d\omega'dk'\frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk
\end{aligned}
$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}} 2\cdot 9 \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|f(k')|}{|i\omega' + \nu|k'|^2|} \\
&\quad |u(k-k', \omega - i\alpha^2 - \omega')| d\omega' dk' \frac{|k|}{|i\omega + \alpha^2 + \nu|k|^2|} d\omega dk \\
&= I \leq \frac{C_1}{\alpha^{\frac{1}{2}}\nu^{\frac{3}{4}}} \int_{\mathbb{R}^3} |f(k)||k|^{-\frac{1}{2}} dk \, |||u|||_\alpha
\end{aligned}$$

and

$$\begin{aligned}
|||T_3 u|||_\alpha &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Bigg| -\hat{\Pi}^S(k) \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} u_m(k-k', \omega - i\alpha^2 - \omega') \\
&\quad \frac{f_n(k')}{i\omega' + \nu|k'|^2} d\omega' dk' \frac{ik_m e_n}{i\omega + \alpha^2 + \nu|k|^2} \Bigg| d\omega dk \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Bigg| -\hat{\Pi}^S(k) \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{f_n(k')}{i\omega' + \nu|k'|^2} \\
&\quad u_m(k-k', \omega - i\alpha^2 - \omega') d\omega' dk' \frac{ik_m e_n}{i\omega + \alpha^2 + \nu|k|^2} \Bigg| d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left| -\hat{\Pi}^S(k) \right| \Bigg| \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{f_n(k')}{i\omega' + \nu|k'|^2} \\
&\quad u_m(k-k', \omega - i\alpha^2 - \omega') d\omega' dk' \frac{ik_m e_n}{i\omega + \alpha^2 + \nu|k|^2} \Bigg| d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left| -\hat{\Pi}^S(k) \right| \sum_{n=1}^{3} \Bigg| \sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{f_n(k')}{i\omega' + \nu|k'|^2} \\
&\quad u_m(k-k', \omega - i\alpha^2 - \omega') d\omega' dk' \frac{ik_m e_n}{i\omega + \alpha^2 + \nu|k|^2} \Bigg| d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left| -\hat{\Pi}^S(k) \right| \sum_{n=1}^{3}\sum_{m=1}^{3} \Bigg| \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{f_n(k')}{i\omega' + \nu|k'|^2} \\
&\quad u_m(k-k', \omega - i\alpha^2 - \omega') d\omega' dk' \frac{ik_m e_n}{i\omega + \alpha^2 + \nu|k|^2} \Bigg| d\omega dk \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left| -\hat{\Pi}^S(k) \right| \sum_{n=1}^{3}\sum_{m=1}^{3} \Bigg| \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{f_n(k')}{i\omega' + \nu|k'|^2} \\
&\quad u_m(k-k', \omega - i\alpha^2 - \omega') d\omega' dk' \Bigg| \left| \frac{ik_m e_n}{i\omega + \alpha^2 + \nu|k|^2} \right| d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left| -\hat{\Pi}^S(k) \right| \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Bigg| \frac{f_n(k')}{i\omega' + \nu|k'|^2} \\
&\quad u_m(k-k', \omega - i\alpha^2 - \omega') \Bigg| d\omega' dk' \frac{|i||k_m||e_n|}{|i\omega + \alpha^2 + \nu|k|^2|} d\omega dk \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left| -\hat{\Pi}^S(k) \right| \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|f_n(k')|}{|i\omega' + \nu|k'|^2|} \\
&\quad |u_m(k-k', \omega - i\alpha^2 - \omega')| d\omega' dk' \frac{|k_m|}{|i\omega + \alpha^2 + \nu|k|^2|} d\omega dk
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} 2\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|f_n(k')|}{|i\omega' + \nu|k'|^2|} \\
&\quad |u_m(k-k', \omega - i\alpha^2 - \omega')| d\omega' dk' \frac{|k_m|}{|i\omega + \alpha^2 + \nu|k|^2|} d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} 2\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|f(k')|}{|i\omega' + \nu|k'|^2|} \\
&\quad |u(k-k', \omega - i\alpha^2 - \omega')| d\omega' dk' \frac{|k|}{|i\omega + \alpha^2 + \nu|k|^2|} d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} 2\cdot 9\int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{|f(k')|}{|i\omega' + \nu|k'|^2|} \\
&\quad |u(k-k', \omega - i\alpha^2 - \omega')| d\omega' dk' \frac{|k|}{|i\omega + \alpha^2 + \nu|k|^2|} d\omega dk \\
&= I \leq \frac{C_1}{\alpha^{\frac{1}{2}}\nu^{\frac{3}{4}}}\int_{\mathbb{R}^3} |f(k)||k|^{-\frac{1}{2}} dk\, |||u|||_\alpha.
\end{aligned}
$$

We establish the following estimate (4.20) using the Residue Theorem. Set

$$
h(\omega') = \frac{1}{i\omega' + \nu|k'|^2}\frac{1}{i(\omega - \omega') + \alpha^2 + \nu|k-k'|^2},
$$

where $\alpha \in \mathbb{R}\setminus\mathbb{R}_-$. The convolution over ω clearly exists because the integral is absolutely convergent by Schwartz inequality. Clearly for every $k' \neq 0$, $k \neq k'$, $\omega \in \mathbb{R}$ there is $M_{k,k',\omega} = 2\max\{\nu|k'|^2, |\omega| + \alpha^2 + \nu|k-k'|^2\}$ and $C > 0$ such that the absolute value

$$
|h(\omega')| \leq C\frac{1}{|\omega'|^2}
$$

for every $\omega' \in \mathbb{C}$, $|\omega'| > M_{k,k',\omega}$. We have lower bound estimates

$$
\begin{aligned}
|\omega'| &> M_{k,k',\omega} = 2\max\{\nu|k'|^2, |\omega| + \alpha^2 + \nu|k-k'|^2\} \\
&> \max\{\nu|k'|^2, |\omega| + \alpha^2 + \nu|k-k'|^2\} \geq \nu|k'|^2 \\
|\omega'| &> M_{k,k',\omega} = 2\max\{\nu|k'|^2, |\omega| + \alpha^2 + \nu|k-k'|^2\} \\
&> \max\{\nu|k'|^2, |\omega| + \alpha^2 + \nu|k-k'|^2\} \geq |\omega| + \alpha^2 + \nu|k-k'|^2 \\
&> |i\omega + \alpha^2 + \nu|k-k'|^2|
\end{aligned}
$$

and upper bound estimates

$$
\begin{aligned}
|i\omega + \alpha^2 + \nu|k-k'|^2| &\leq \max\{\nu|k'|^2, |\omega| + \alpha^2 + \nu|k-k'|^2\} \\
&= \frac{1}{2}M_{k,k',\omega} \leq \frac{1}{2}|\omega'| \\
\nu|k'|^2 &\leq \max\{\nu|k'|^2, |\omega| + \alpha^2 + \nu|k-k'|^2\} \\
&= \frac{1}{2}M_{k,k',\omega} \leq \frac{1}{2}|\omega'|.
\end{aligned}
$$

Hence we can estimate

$$
|h(\omega')| = \left|\frac{1}{i\omega' + \nu|k'|^2}\frac{1}{i(\omega - \omega') + \alpha^2 + \nu|k-k'|^2}\right|
$$

$$
\begin{aligned}
&= \frac{1}{|i\omega' + \nu|k'|^2|}\frac{1}{|i(\omega-\omega') + \alpha^2 + \nu|k-k'|^2|}\\
&= \frac{1}{|i\omega' + \nu|k'|^2|}\frac{1}{|-i\omega' + i\omega + \alpha^2 + \nu|k-k'|^2|}\\
&\le \frac{1}{||i\omega'| - |\nu|k'|^2||}\frac{1}{||-i\omega'| - |i\omega + \alpha^2 + \nu|k-k'|^2||}\\
&= \frac{1}{||\omega'| - \nu|k'|^2||}\frac{1}{||\omega'| - |i\omega + \alpha^2 + \nu|k-k'|^2||}\\
&= \frac{1}{|\omega'| - \nu|k'|^2}\frac{1}{|\omega'| - |i\omega + \alpha^2 + \nu|k-k'|^2|}\\
&\le \frac{1}{|\omega'| - \frac{1}{2}|\omega'|}\frac{1}{|\omega'| - \frac{1}{2}|\omega'|} = 4\frac{1}{|\omega'|^2} = C\frac{1}{|\omega'|^2}.
\end{aligned}
$$

We take rectangles with bottom $[-M, M]$ on the real axis and top on the line that intersects iM and is parallel to the real axis. We set $n = 1$ and $z_1 = i\nu|k'|^2$. Taking limits of integrals, when M tends to infinity, we notice that integrals over those sides of the rectangle, that are not on the real axis, tend to zero. Hence we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{i\omega' + \nu|k'|^2}\frac{1}{i(\omega-\omega') + \alpha^2 + \nu|k-k'|^2}\,d\omega' = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\,[h, z_k]\\
&= 2\pi i \lim_{\omega' \to i\nu|k'|^2} (\omega' - i\nu|k'|^2)\frac{1}{i\omega' + \nu|k'|^2}\,\frac{1}{i(\omega-\omega') + \alpha^2 + \nu|k-k'|^2}\\
&= \frac{2\pi}{i\omega + \alpha^2 + \nu|k'|^2 + \nu|k-k'|^2}.
\end{aligned}
$$

We calculate further

$$
\begin{aligned}
|k'|^2 + |k-k'|^2 &= |k'|^2 + |k'-k|^2 = |k' - \frac{1}{2}k + \frac{1}{2}k|^2 + |k' - \frac{1}{2}k - \frac{1}{2}k|^2\\
&= (k' - \frac{1}{2}k)\cdot(k' - \frac{1}{2}k) + 2(k' - \frac{1}{2}k)\cdot\frac{1}{2}k + \frac{1}{2}k\cdot\frac{1}{2}k\\
&\quad +(k' - \frac{1}{2}k)\cdot(k' - \frac{1}{2}k) - 2(k' - \frac{1}{2}k)\cdot\frac{1}{2}k + \frac{1}{2}k\cdot\frac{1}{2}k\\
&= 2|k' - \frac{1}{2}k|^2 + \frac{1}{2}|k|^2,
\end{aligned}
$$

$$
\begin{aligned}
\nu|k'|^2 + \nu|k-k'|^2 &= \nu(|k'|^2 + |k-k'|^2) = \nu(2|k' - \frac{1}{2}k|^2 + \frac{1}{2}|k|^2)\\
&= \nu\cdot 2|k' - \frac{1}{2}k|^2 + \nu\cdot\frac{1}{2}|k|^2\\
&= 2\nu|k' - \frac{1}{2}k|^2 + \frac{1}{2}\nu|k|^2
\end{aligned}
$$

and estimate the convolution integral above. We obtain

$$
\begin{aligned}
&\left|\int_{-\infty}^{\infty} \frac{1}{i\omega' + \nu|k'|^2}\frac{1}{i(\omega-\omega') + \alpha^2 + \nu|k-k'|^2}\,d\omega'\right|\\
&= \frac{2\pi}{|i\omega + \alpha^2 + \nu|k'|^2 + \nu|k-k'|^2|}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2\pi}{|i\omega + \alpha^2 + 2\nu|k' - \frac{1}{2}k|^2 + \frac{1}{2}\nu|k|^2|} \leq \frac{2\pi}{|i\omega + \alpha^2 + \frac{1}{2}\nu|k|^2|} \\
&\leq \frac{2\pi}{|\frac{1}{2}i\omega + \alpha^2 + \frac{1}{2}\nu|k|^2|} \leq \frac{2\pi}{|\frac{1}{2}i\omega + \frac{1}{2}\alpha^2 + \frac{1}{2}\nu|k|^2|} \\
&= \frac{4\pi}{|i\omega + \alpha^2 + \nu|k|^2|}.
\end{aligned}
$$

The function g in (4.18) is integrable and its α-norm decreases to zero as α tends to infinity, which is shown by the following estimate. We recall that $f \in \mathcal{S}_3^3$, which implies that f is rapidly decreasing, and change the order of integration by Fubini's theorem (Theorem 8.8 in [11]) to obtain

$$
\begin{aligned}
|||g|||_\alpha &= \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Bigg| -\hat{\Pi}^S(k) \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{f_m(k')}{i\omega' + \nu|k'|^2} \\
&\qquad \frac{f_n(k-k')}{i(\omega-\omega') + \alpha^2 + \nu|k-k'|^2} d\omega' dk' \frac{ik_m e_n}{i\omega + \alpha^2 + \nu|k|^2} \Bigg| d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Big| -\hat{\Pi}^S(k) \Big| \Bigg| \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{f_m(k')}{i\omega' + \nu|k'|^2} \\
&\qquad \frac{f_n(k-k')}{i(\omega-\omega') + \alpha^2 + \nu|k-k'|^2} d\omega' dk' \frac{ik_m e_n}{i\omega + \alpha^2 + \nu|k|^2} \Bigg| d\omega dk \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Big| -\hat{\Pi}^S(k) \Big| \Bigg| \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3} f_m(k') f_n(k-k') \int_{\mathbb{R}} \frac{1}{i\omega' + \nu|k'|^2} \\
&\qquad \frac{1}{i(\omega-\omega') + \alpha^2 + \nu|k-k'|^2} d\omega' dk' \frac{ik_m e_n}{i\omega + \alpha^2 + \nu|k|^2} \Bigg| d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Big| -\hat{\Pi}^S(k) \Big| \sum_{n=1}^{3} \Bigg| \sum_{m=1}^{3} \int_{\mathbb{R}^3} f_m(k') f_n(k-k') \int_{\mathbb{R}} \frac{1}{i\omega' + \nu|k'|^2} \\
&\qquad \frac{1}{i(\omega-\omega') + \alpha^2 + \nu|k-k'|^2} d\omega' dk' \frac{ik_m e_n}{i\omega + \alpha^2 + \nu|k|^2} \Bigg| d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Big| -\hat{\Pi}^S(k) \Big| \sum_{n=1}^{3}\sum_{m=1}^{3} \Bigg| \int_{\mathbb{R}^3} f_m(k') f_n(k-k') \int_{\mathbb{R}} \frac{1}{i\omega' + \nu|k'|^2} \\
&\qquad \frac{1}{i(\omega-\omega') + \alpha^2 + \nu|k-k'|^2} d\omega' dk' \frac{ik_m e_n}{i\omega + \alpha^2 + \nu|k|^2} \Bigg| d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Big| -\hat{\Pi}^S(k) \Big| \sum_{n=1}^{3}\sum_{m=1}^{3} \Bigg| \int_{\mathbb{R}^3} f_m(k') f_n(k-k') \int_{\mathbb{R}} \frac{1}{i\omega' + \nu|k'|^2} \\
&\qquad \frac{1}{i(\omega-\omega') + \alpha^2 + \nu|k-k'|^2} d\omega' dk' \Bigg| \Bigg| \frac{ik_m e_n}{i\omega + \alpha^2 + \nu|k|^2} \Bigg| d\omega dk \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}} \Big| -\hat{\Pi}^S(k) \Big| \sum_{n=1}^{3}\sum_{m=1}^{3} \int_{\mathbb{R}^3} \Bigg| f_m(k') f_n(k-k') \int_{\mathbb{R}} \frac{1}{i\omega' + \nu|k'|^2} \\
&\qquad \frac{1}{i(\omega-\omega') + \alpha^2 + \nu|k-k'|^2} d\omega' \Bigg| dk' \frac{|i||k_m||e_n|}{|i\omega + \alpha^2 + \nu|k|^2|} d\omega dk
\end{aligned}
$$

$$\begin{aligned}
\leq\ & \int_{\mathbb{R}^3}\int_{\mathbb{R}}\left|-\hat{\Pi}^S(k)\right|\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\left|f_m(k')\right|\left|f_n(k-k')\right|\left|\int_{\mathbb{R}}\frac{1}{i\omega'+\nu|k'|^2}\right.\\
& \left.\frac{1}{i(\omega-\omega')+\alpha^2+\nu|k-k'|^2}d\omega'\right|dk'\,\frac{|k_m|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk\\
\leq\ & \int_{\mathbb{R}^3}\int_{\mathbb{R}}\left|-\hat{\Pi}^S(k)\right|\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}|f_m(k')||f_n(k-k')|\\
& \frac{4\pi}{|i\omega+\alpha^2+\nu|k|^2|}dk'\,\frac{|k_m|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk\\
\leq\ & \int_{\mathbb{R}^3}\int_{\mathbb{R}}\left|-\hat{\Pi}^S(k)\right|\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}|f(k')||f(k-k')|\\
& \frac{4\pi}{|i\omega+\alpha^2+\nu|k|^2|}dk'\,\frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}de\omega dk\\
\leq\ & \int_{\mathbb{R}^3}\int_{\mathbb{R}}2\sum_{n=1}^{3}\sum_{m=1}^{3}\int_{\mathbb{R}^3}|f(k')||f(k-k')|\frac{4\pi}{|i\omega+\alpha^2+\nu|k|^2|}dk'\\
& \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk\\
=\ & \int_{\mathbb{R}^3}\int_{\mathbb{R}}2\cdot 9\int_{\mathbb{R}^3}|f(k')||f(k-k')|\frac{4\pi}{|i\omega+\alpha^2+\nu|k|^2|}dk'\\
& \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}de\omega dk\\
\leq\ & 72\pi\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k')||f(k-k')|\frac{1}{|i\omega+\alpha^2+\nu|k|^2|}dk'\\
& \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk\\
=\ & 72\pi\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{1}{|i\omega+\alpha^2+\nu|k|^2|}|f(k')||f(k-k')|dk'\\
& \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk\\
=\ & 72\pi\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{1}{|i\omega+\alpha^2+\nu|k|^2|}\int_{\mathbb{R}^3}|f(k')||f(k-k')|dk'\\
& \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk\\
=\ & 72\pi\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k')||f(k-k')|dk'\frac{1}{|i\omega+\alpha^2+\nu|k|^2|}\\
& \frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|}d\omega dk\\
=\ & 72\pi\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|f(k')||f(k-k')|dk'\frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|^2}d\omega dk\\
=\ & 72\pi\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|f(k')||f(k-k')|dk'\int_{\mathbb{R}}\frac{|k|}{|i\omega+\alpha^2+\nu|k|^2|^2}d\omega dk
\end{aligned}$$

$$
\begin{aligned}
&= 72\pi \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(k')||f(k-k')|dk' \int_{\mathbb{R}} \frac{|k|}{\omega^2+(\alpha^2+\nu|k|^2|)^2} d\omega dk \\
&= 72\pi \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(k')||f(k-k')|dk' \frac{|k|}{\alpha^2+\nu|k|^2} \int_{\mathbb{R}} \frac{1}{\omega'^2+1} d\omega' dk \\
&= 72\pi \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(k')||f(k-k')|dk' \frac{\pi|k|}{\alpha^2+\nu|k|^2} dk \\
&= 72\pi^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(k')||f(k-k')|dk' \frac{|k|}{\alpha^2+\nu|k|^2} dk \\
&\leq 72\pi^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(k')||f(k-k')|dk' \frac{1}{\alpha\sqrt{\nu}} dk \\
&\leq \frac{72\pi^2}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(k')||f(k-k')|dk' dk \\
&= \frac{72\pi^2}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(k')||f(k-k')|dk dk' \\
&= \frac{72\pi^2}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3} |f(k')| \int_{\mathbb{R}^3} |f(k-k')|dk dk' \\
&= \frac{72\pi^2}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3} |f(k')| \int_{\mathbb{R}^3} |f(k)|dk dk' \\
&= \frac{72\pi^2}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(k)|dk|f(k')|dk' \\
&= \frac{72\pi^2}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3} |f(k)|dk \int_{\mathbb{R}^3} |f(k')|dk' \\
&= \frac{72\pi^2}{\alpha\sqrt{\nu}} \int_{\mathbb{R}^3} |f(k')|dk' \int_{\mathbb{R}^3} |f(k)|dk = \frac{72\pi^2}{\alpha\sqrt{\nu}} ||f||_1^2. \qquad (4.20)
\end{aligned}
$$

We see that the sum $T_2 + T_3$ of T becomes contractive for large values of α. In addition $|||g|||_\alpha \to 0$, when $\alpha \to \infty$. Boundedness of T_1 was not shown. Numerical experiments support the theory that T_1 is unbounded in norm (4.19), which allows divergent iteration with too big initial values. The theory applies equation (4.18) in the lower complex plane. However, the transformed operator $\frac{ik}{i\omega+\nu|k|^2}$ was analyzed successfully. Also linearization of T_1, where one of convolution factors is replaced by constant function, can be easily proved to be contractive, which allows to show the existence of a solution for corresponding linearized equation.

Chapter 5

Numerical experiment

The aim of this part of the work is to derive a numerical method for computation of incompressible flow. The method is written for the two-dimensional equation

$$\begin{aligned} v(x,t) &= \int_{\mathbb{R}^n} v_0(x-x')\frac{1}{\sqrt{4\pi\nu t}^n}e^{-\frac{|x'|^2}{4\nu t}}dx' \\ &+ \int_{\mathbb{R}^n}\int_0^t -P_2(v\cdot\nabla)v(x-x',t-\tau)\frac{1}{\sqrt{4\pi\nu\tau}^n}e^{-\frac{|x'|^2}{4\nu\tau}}d\tau dx', \end{aligned} \quad (5.1)$$

where $n = 2$, that is, for the equation (2.28). The only highly successful thing was the computation of the projection P_2, that is implemented using equations (2.13) and (2.14). Differential operators were defined in an orthogonal and uniform grid by their difference approximations and they were calculated using their Fourier-domain representations, where Fourier transform was replaced by DFT.

5.1 Motivation for numerical experiment

In my mathematics special assingment [12] I discovered the following way to solve the Dirichlet problem

$$\left\{ \begin{array}{lcll} \Delta u &=& 0 & \Omega \\ u &=& f & \partial\Omega \end{array} \right. , \quad (5.2)$$

where we assume that the boundary is smooth, that is, it has a smooth periodic parametric representation $r : \mathbb{R} \to \mathbb{R}^2$. Definitions

$$\begin{aligned} \Lambda(\varphi) &= \int_{\partial\Omega} \varphi d\sigma, \\ (\psi\Lambda)(\varphi) &= \Lambda(\psi\varphi) \end{aligned} \quad (5.3)$$

imply

$$\begin{aligned} u\Lambda &= f\Lambda \\ w * u\Lambda &= w * f\Lambda, \end{aligned} \quad (5.4)$$

where $-w$ is the Poisson kernel. Now definitions

$$\begin{aligned} Tu &= w * u\Lambda, \\ g &= w * f\Lambda \end{aligned}$$

lead to the equation

$$Tu = g, \tag{5.5}$$

where T is symmetric and positive semidefinite. I applied fixed point iteration

$$u^{n+1} = u^n - \alpha(Tu^n - g). \tag{5.6}$$

to the equation (5.5) with a suitably small $\alpha > 0$. The method (5.6) for problem (5.2) was slow, but it applied numerical representation efficiently. The code applied an iteration loop and memory allocation status was designed to remain unchanged during iteration. Error decreased to numerical noise defined by digital representation specifications.

5.2 Iteration method and its behaviour

I decided to try the method (5.6) for (5.1) and wrote

$$u^{n+1} = u^n - \alpha(u^n - g - Tu^n),$$

where

$$\begin{aligned} Tv(x,t) &= \int_0^t \int_{\mathbb{R}^2} -P_2(v \cdot \nabla)v(x-x', t-t')\, w(x',t')\, dx'dt', \quad t > 0, \\ Tv(x,0) &= 0, \end{aligned}$$

$$\begin{aligned} g(x,t) &= \int_{\mathbb{R}^2} v_0(x-x')w(x',t)dx', \quad t > 0, \\ g(x,0) &= v_0(x). \end{aligned}$$

Here w is the heat kernel with the expression

$$w(x,t) = \frac{1}{\sqrt{4\pi\nu t}^2} e^{-\frac{|x|^2}{4\nu t}}, \quad t > 0.$$

Recall that projections P_1 and P_2 have expressions

$$\begin{aligned} P_1 v &= \nabla\Delta^{-1}\nabla \cdot v, \\ P_2 v &= v - P_1 v. \end{aligned}$$

Initial guess $u^0 = g$ was applied instead of $u^0 = 0$. The iteration (5.7) behaved similarly to the iteration (5.6), that is, it seemed to behave in a favourable way with small α:s and blow up for large α:s. Residual error decrease seemed to be linear and not superlinear in both cases.

5.3 Source code and results

Numerical experiment applied the code below.

```
function state = code()

  DATA = '/home/jvihtane/tests/test4/data';
  CODE = '/home/jvihtane/tests/test4/code';

  N = 2^6-1;

  % Definition of discrete partial differentiation

  dx = zeros(N,N,N);  dx(2,1,1) = -1;  dx(N,1,1) = 1;
  dy = zeros(N,N,N);  dy(1,2,1) = -1;  dy(1,N,1) = 1;
  dt = zeros(N,N,N);  dt(1,1,2) = -1;  dt(1,1,N) = 1;

  % Definition of discrete laplace operator

  D = ifft3(fft3(dx).*fft3(dx)+fft3(dy).*fft3(dy));
  D0 = 1/N^3*ones(N,N,N) + D;

  % Smoothening functions

  phix = zeros(N,N,N);  phix(N,1,1) = 1;  phix(1,1,1) = 2;  phix(2,1,1) = 1;
  phiy = zeros(N,N,N);  phiy(1,N,1) = 1;  phiy(1,1,1) = 2;  phiy(1,2,1) = 1;
  phit = zeros(N,N,N);  phit(1,1,N) = 1;  phit(1,1,1) = 2;  phit(1,1,2) = 1;

  phi = ifft3(fft3(phix).*fft3(phiy).*fft3(phit));

  % Cut functions that prevent periodic disturbance over time
  % X0 is suitable for Volterra kernal convolution and is applied for heat kernel convolution

  X0 = zeros(N,N,N);  X0(:,:,3:(N-5)/2) = ones(N,N,(N-9)/2);   t0 = zeros(N,N,N);
  X1 = zeros(N,N,N);  X1(:,:,2:(N-3)/2) = ones(N,N,(N-5)/2);   t0(:,:,1) = ones(N,N);

  % Construction of zero mean data with values = c_1 on a line and values = c_2 outside of the line

  data = zeros(N,N,N);  data(1:8,1,1) = 1;  data(N-6:N,1,1) = 1;
  data = ifft3(fft3(phi).*fft3(phi).*fft3(data)).*t0;
  data = data - sum(sum(sum(data)))/N^3;

  % Definition of heat kernel

  d = 1/N^3*ones(N,N,N);  d(1,1,1) = 1-1/N^3;
  w = ifft3(fft3(d)./fft3(dt-1/6*D0));

  % Transformations of partial differential, laplace and heat operators, transformations of derivatives of the
  % laplace operator

  fft3dx = fft3(dx);  fft3D0 = fft3(D0);   fft3dx_fft3D0 = fft3dx./fft3D0;
  fft3dy = fft3(dy);  fft3w  = fft3(w);    fft3dy_fft3D0 = fft3dy./fft3D0;
  fft3dt = fft3(dt);                       fft3dt_D0     = fft3dt-fft3D0;

  % Iteration parameter

  alpha = 1e-1;

  % Differentiation of data <data>, definition of x- and y-conponents of the initial data u_0

  u0x  =  1/85*ifft3(fft3(w).*fft3(dy).*fft3(data));  ux = u0x;  ux(:,:,(N-1)/2:end) = 0;  resx = zeros(N,N,N);
  u0y  = -1/85*ifft3(fft3(w).*fft3(dx).*fft3(data));  uy = u0y;  uy(:,:,(N-1)/2:end) = 0;  resy = zeros(N,N,N);

  for i=1:2^14

    % Field F = (Fx,Fy)
    % Sybbol d = \cdot \nabla (does not override the definition of partial differential operators)
    % DP is longitudinal projection times 1/38
    % CP is transversal projection times 1/38

    udux  =  ux.*ifft3(fft3dx.*fft3(ux))+uy.*ifft3(fft3dy.*fft3(ux));
    uduy  =  ux.*ifft3(fft3dx.*fft3(uy))+uy.*ifft3(fft3dy.*fft3(uy));

    % Longitudinal projection (DPudux,DPuduy) and transversal projection (CPudux,CPuduy) of the field (ux,uy)

    DPudux  =  ifft3(fft3dx_fft3D0.*(fft3dx.*fft3(udux)+fft3dy.*fft3(uduy)));   CPudux  =  udux - DPudux;
    DPuduy  =  ifft3(fft3dy_fft3D0.*(fft3dx.*fft3(udux)+fft3dy.*fft3(uduy)));   CPuduy  =  uduy - DPuduy;

    % Residual of the field (ux,uy), next iteration element

    resx  =  (ifft3(fft3(ux)./fft3w)+CPudux).*X0;    ux = ux - alpha*(ux-u0x+ifft3(fft3w.*fft3(CPudux.*X0))).*X1;
    resy  =  (ifft3(fft3(uy)./fft3w)+CPuduy).*X0;    uy = uy - alpha*(uy-u0y+ifft3(fft3w.*fft3(CPuduy.*X0))).*X1;

    % Residual norm calculations and database update

    resv = sqrt(sum(sum(resx.*conj(resx)+resy.*conj(resy))));
    err(i) = log(sqrt(sum(sum(sum(resv.*conj(resv))))));
    fprintf(1,'%e\n',exp(err(i)));

    if i/16 == floor(i/16)
      data1  =  ux;  data3 = resv;
```

```
    data2 = uy; data4 = err;

    fprintf(1,'Writing ...\n');
    cd(DATA);
    save data9.mat data1 data2 data3 data4
    cd(CODE);
    fprintf(1,'Done.\n');
  end

end
fprintf(1,'\n');

state = 0;



function state = viewdata9()

DATA = '/home/jvihtane/tests/test4/data';
CODE = '/home/jvihtane/tests/test4/code';

i = 256+1:256+256;
map=[i;i;i]'/(256+256);

cd(DATA);
load data9.mat
cd(CODE);

who
size(data1)
size(data2)
size(data3)
size(data4)

N = length(data1);

phit = zeros(N,N,N); phit(1,1,N) = 1; phit(1,1,1) = 2; phit(1,1,2) = 1;

data1 = real(ifft3(fft3(phit).*fft3(data1)));
data2 = real(ifft3(fft3(phit).*fft3(data2)));

figure(3);
plot(data4);
print -deps residual9.eps

v = permute(data3,[3,1,2]);

figure(4);
plot(v(1:ceil(N/2-1)))

for k=1:(N-5)/2

  m1 = max(max(abs(real(fftshift(data1(:,:,1*(k-1)+1))))));
  m2 = max(max(abs(real(fftshift(data2(:,:,1*(k-1)+1))))));
  fprintf(1,'%d\n',k-1);

  if mod(k,10) == 1
    p = (k-1)/10;
    q = ((N-5)/2-mod((N-5)/2,10))/10+1;

    figure(5); subplot(2,q,p+1); colormap(map);
    imagesc(transpose(real(fftshift(data1(:,:,1*(k-1)+1)))),[-m1 m1]);
    xlabel(strcat('Maximum M = ',num2str(m1))); pbaspect([2,2,1]); axis xy;

    figure(5); subplot(2,q,p+q+1); colormap(map);
    imagesc(transpose(real(fftshift(data2(:,:,1*(k-1)+1)))),[-m2 m2]);
    xlabel(strcat('Maximum M = ',num2str(m2))); pbaspect([2,2,1]); axis xy;

  end

  figure(1); colormap(map);
  imagesc(transpose(real(fftshift(data1(:,:,1*(k-1)+1)))),[-m1 m1]);
  title(strcat('Maximum M = ',num2str(m1)));
  axis xy; axis square;

  figure(2); colormap(map);
  imagesc(transpose(real(fftshift(data2(:,:,1*(k-1)+1)))),[-m2 m2]);
  title(strcat('Maximum M = ',num2str(m2)));
  axis xy; axis square;

  drawnow;

end

figure(5); print -deps field9.eps

state = 0;
```

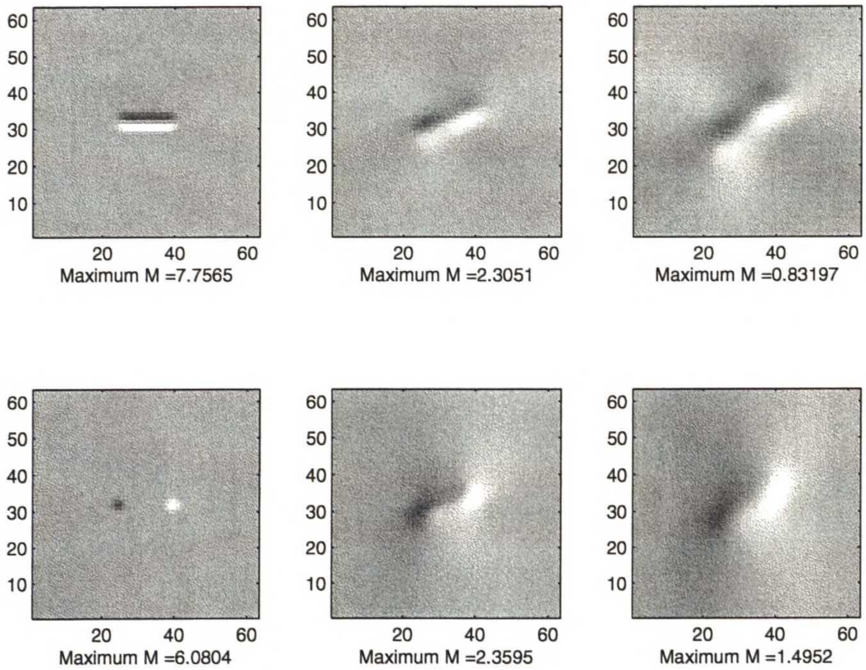

Figure 5.1: Field components x and y at t=0, t=10 and t=20. Space unit is meter m, time unit second s and velocity unit meters per second m/s.

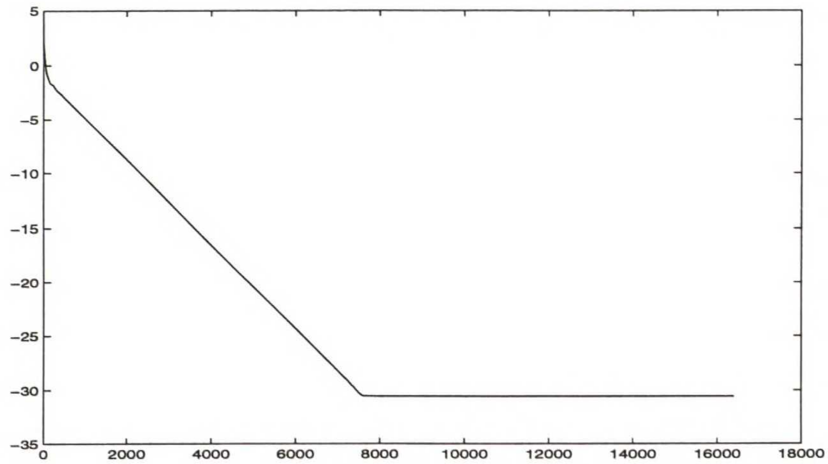

Figure 5.2: Square integral norm of residual on logarithmic scale (natural logarithm)

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